Transient Chaos in an Elasto-Plastic Beam with Hardening

M. A. Savi
Department of Mechanical Engineering
PEM/COPPE/UFRJ
21945-970 Rio de Janeiro, RJ. Brazil
savi@serv.com.ufrj.br

P. M. C. L. Pacheco
Department of Mechanical Engineering
CEFET/RJ
20.271.110 Rio de Janeiro, RJ. Brazil
calas@cefet-rj.br

This contribution discusses the transient chaos on a pin-ended elasto-plastic beam with both kinematic and isotropic hardening. An iterative numerical procedure, based on the operator split technique, is developed. Numerical simulations show that transient chaos causes sensitivity on initial conditions, which imply that the system response may become unpredictable.
Keywords*: Chaos, non-linear dynamics, elasto-plasticity, hardening*

Introduction

The study of elasto-plastic structures is important in many engineering problems. It is a non-linear phenomenon and its dynamical response is very rich. Shanley (1947) proposes a model where a pin-ended beam is represented by two rigid links joined by an elasto-plastic element, referred as cell. This element has two short flanges. Each one is elastic-perfectly plastic. Hardening effect is not considered. Symonds and Yu (1985) studied the problem either by finite element method or by employing a simplified Shanley model. Poddar *et al.* (1988) consider the chaotic behavior of the Symonds' model when it is periodically excited by a series of positive and negative impulses. Pratap *et al.* (1994a,b) present a new model where the cell is replaced by an elasto-plastic torsional spring. Kinematic hardening is considered to describe the constitutive behavior of this spring.

The present work revisits the Symonds' model by considering hardening effect. This effect represents the way of how plastic strains modify the yield surface. It is a usual situation and may happen in many different ways, depending specifically on the material and on the loading history. There are many idealized models to describe this phenomenon, which can usually be properly represented by a combination of kinematic and isotropic hardening (Lemaitre and Chaboche, 1990). This article considers both kinematic and isotropic hardening on the model. Small displacement hypothesis, considered on all the cited models, is not assumed here. An iterative numerical procedure is developed to solve the equations of motion. Numerical investigations are done giving special attention on chaotic response.

Model for an Elasto-Plastic Beam

Symonds' model describes beam behavior by considering a pin-ended beam with length $2L$, and uniform rectangular cross section of area $A = bh'$. The beam is represented by two rigid links, each of length L, joined by an elasto-plastic element. The two rigid bars are assumed to have mass per unit length ρ, the same as for the uniform beam (Shanley, 1947). The beam model is depicted in Fig.1.

Geometric considerations permit to define the relation between the cell position, a, and the angle of rotation, φ, and also the semi-extension of the cell centerline, e, and the semi-extension of each flange, e_1 and e_2,

$$\begin{aligned} a &= L\sin\varphi \\ e &= L(\cos\varphi_0 - \cos\varphi) \\ e_1 &= e + \frac{h}{2}\sin\varphi \\ e_2 &= e - \frac{h}{2}\sin\varphi \ . \end{aligned} \qquad (1)$$

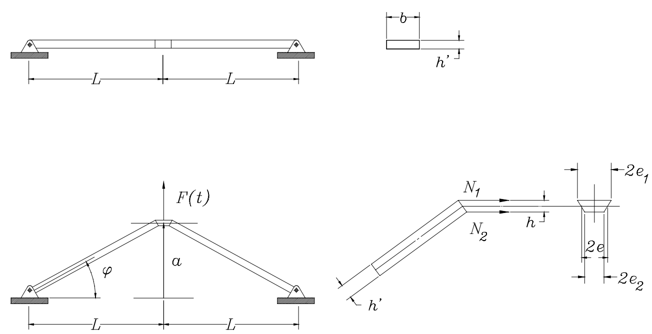


Figure 1. Elasto-plastic Beam.

The constitutive equation considers a linear kinematic and isotropic hardening, as presented in Savi and Pacheco (1997),

$$\begin{aligned} \sigma &= E(\varepsilon - \varepsilon^p) \\ \dot{\varepsilon}^p &= \gamma\ \mathrm{sign}(\sigma - \beta) \\ \dot{\alpha} &= \gamma \\ \dot{\beta} &= \gamma\ H\ \mathrm{sign}(\sigma - \beta)\ , \end{aligned} \qquad (2)$$

where $\mathrm{sign}(\sigma) = \sigma / |\sigma|$. σ is the one-dimensional stress, ε and ε^p are the total and plastic one-dimensional strain, respectively. γ represents the rate at which plastic deformations take place. E is the Young modulus. The yield function, $h(\sigma,\alpha,\beta)$, the Kuhn-Tucker conditions and the consistency condition are given by (Savi and Pacheco, 1997):

$$\begin{aligned} h(\sigma,\alpha,\beta) &= |\sigma - \beta| - (\sigma_y + K\alpha) \\ \gamma &\geq 0 \\ \gamma\, h(\sigma,\alpha,\beta) &= 0 \\ \gamma\, \dot{h}(\sigma,\alpha,\beta) &= 0 \quad \text{if} \quad h(\sigma,\alpha,\beta) = 0\ . \end{aligned} \qquad (3)$$

Here, K is the plastic modulus. The yield function shows that kinematic hardening causes the elastic domain translation, while isotropic hardening causes the expansion of this domain.

Presented at COBEM 99 – 15th Brazilian Congress of Mechanical Engineering, 22-26 November 1999, São Paulo. SP. Brazil.
Paper accepted: July, 2002. Technical Editor: José Roberto de França Arruda.

The force and moment resultants in the cell are taken by considering the same relations of those of a sandwich beam, consisting of two bars each in simple tension or compression (Symonds and Yu, 1985). Hence,

$$N = N_1 + N_2 = \int_{A_1} \sigma_1 dA + \int_{A_2} \sigma_2 dA \ , \quad M = (N_2 - N_1)\frac{h}{2} \qquad (4)$$

where N_1 and N_2 are the forces on each flange. Assuming that the total strain on each flange is obtained by dividing the semi-extension by the semi-length of the beam, and the area of each flange is a half of the beam cross section area, hence,

$$N_i = \frac{EA}{2L}(e_i - e_i^p) \ \ (i = 1,2). \qquad (5)$$

where e_i^p is the plastic semi-extension on each flange.

In order to obtain the governing equations of the model, one establishes the equilibrium of moment on the half beam. Neglecting the inertia of the elasto-plastic element, and assuming a linear viscous external dissipation, one obtains

$$\ddot{\varphi} + \mu c\,\dot{\varphi} + \mu L\,N \sin\varphi - \mu M = \frac{\mu L}{2}F(t) \qquad (6)$$

where $\mu = 3/\rho L^3$ and c is the linear viscous dissipation parameter. N and M are given by equations (1,5). Now, consider the following definitions,

$$\omega_0^2 = \mu L\,N_y \ , \quad c_0 = \mu c/\omega_0 \ , \quad \mu_0 = M_y / L\,N_y \ , \quad f = F/2N_y \ ,$$
$$\tau = \omega_0 t$$
$$n = N/N_y \ , \ m = M/M_y \, . \qquad (7)$$

Here $N_y = \sigma_y A$ and $M_y = N_y\,h/2$. Denoting the non-dimensional time derivative by $(\ )' = d(\ )/d\tau$, the following system can be written,

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -c_0 y_2 - n \sin y_1 + \mu_0\, m + f(\tau) \end{aligned} \qquad (8)$$

Actually, the space of state variables includes more variables than y_1 and y_2 (Poddar *et al.*, 1988), however, the analysis is developed on a subspace of dimension 2 (Savi and Pacheco, 1997). The numerical solution procedure here proposed uses the operator split technique (Ortiz *et al.*, 1983). First, the equations of motion (8) are integrated using any classical scheme, like fourth order Runge-Kutta, assuming that variables n and m are known parameters. Hence, n and m are evaluated by considering an elastic predictor step (trial state), where plastic variables ($\varepsilon^p,\alpha,\beta$) remain constant from the previous time instant. The next step of solution procedure consists on a plastic corrector step where the feasibility of trial state is evaluated using the return mapping algorithm (Simo and Taylor, 1985). An iterative process takes place until the convergence is achieved (Savi and Pacheco, 1997).

Transient Chaos

This section presents the beam response under harmonic sinusoidal excitation, $f(\tau) = \delta \sin(\Omega\tau)$. For all numerical simulations presented in this article one has taken L = 0.10m, b = 0.02m, h' = 0.04m, $h = 0.68h'$, ρ = 0.216 kg/m, E = 120 GPa, σ_y = 0.3 GPa, K = 0.10 GPa, H = 0.44 GPa. The procedure converges with time steps smaller than $\Delta\tau = 2\pi/1000\Omega$.

Savi and Pacheco (1997) show that an elasto-plastic oscillator with isotropic hardening tends to an elastic steady state response. Therefore, similar behavior may be expected to the beam response and, after a transient response, an elastic steady state response is reached where plastic variables remain constant.

In order to start the analysis, one considers bifurcation diagrams which presents stroboscopically sampled angular displacement values, y_1, under the slow quasi-static increase of the driving force amplitude, δ. The dissipation parameter is $c_0 = 1.5$ and the first 30 cycles are neglected. Different frequency parameters are considered. Regions with clouds of points, usually associated to chaotic behavior, and jumps, can be observed in Fig.2. For the present dynamical system, the clouds of points are associated with transient behavior.

Firstly, one analyzes the jump phenomenon, which is associated with changes in global behavior. As it can be seen on bifurcation diagrams, small changes in force amplitudes cause great variations in the steady state responses. When Ω = 1, jump is near δ = 0.25. For Ω = 2, jump does not occur for the range of forcing parameter considered. On the other hand, for Ω = 0.75, jump is near δ = 0.18, but only in transient behavior. Figure 3 shows steady state response for Ω = 1 and two forcing amplitudes very close (δ = 0.24 and δ = 0.25). There is a qualitative change on the global dynamics.

In order to discuss transient chaos, Fig.4 shows the Poincaré section when δ = 0.3 and Ω = 0.75. Figure 4 presents a transient strange attractor where, after 1300 cycles, the motion tends to a single point meaning a periodic response. The Fast Fourier Transform (FFT) analysis permits to identify clearly the difference between the two responses. When transient chaos is taking place, the FFT presents continuous spectra, where the energy is spread over a wider bandwidth (Fig.5a). On the other hand, after this transient, the response becomes periodic, and the FFT presents discrete spectra where a finite number of frequencies contribute for the response (Fig.5b).

Chaotic motion presents high sensitivity on initial conditions and, in spite of the periodic steady state behavior, transient chaos causes unpredictability on the motion. By considering different initial conditions, very close from the previous case, different steady state responses may be observed. Figure 6 illustrates this behavior. The previous example considers (y_1, y_2) = (0,0), and other situations are analyzed for (y_1, y_2) = (0,-1e-4), (y_1, y_2) = (0,+1e-4), (y_1, y_2) = (-2.5e-4,0) and (y_1, y_2) = (+2.5e-4,0). These conditions represent variations less than 0.1% of the maximum angular displacement and velocity at steady state. Hence, transient chaos, associated with fractal basin boundaries, may cause practical problems in predicting the beam response (Poddar *et al.*, 1988).

Grebogi *et al.* (1983) defines crises phenomena as a collision between a chaotic attractor and a coexisting unstable fixed point or periodic orbit. In this situation, chaotic behavior appears or disappears for some parameter change. In transient chaos, after a finite time, the orbit rapidly leaves the chaotic region, establishing a periodic or quasiperiodic motion (Moon, 1992). The term transient chaos is used here to describe a chaotic like response, which will become periodic after some cycles. A particular set of physical parameters is considered. This behavior may be understood as crises of the system where the parameters are represented by the internal variables associated with plastic behavior, whose values change with time.

Transient chaos also occurs when Ω = 1. Figure 7 shows the Poincaré section for δ = 0.5, and again, it is possible to observe a transient strange attractor. After 540 cycles, however, this motion tends to a single point meaning periodic response. Figure 8 shows the difference between the two kinds of behavior by considering the FFT analysis. Sensitivity on initial conditions is different from the previous example. Assuming the preceding variations on initial conditions, the beam does not alter steady state response.

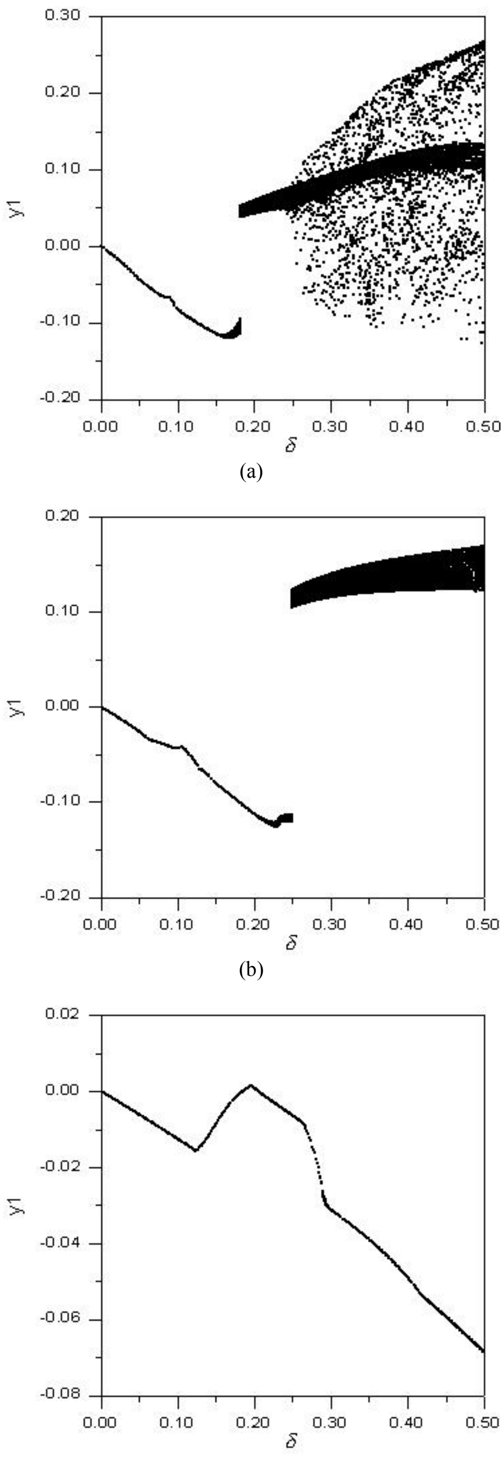


Figure 2. Bifurcation Diagrams. (a) $\Omega = 0.75$; (b) $\Omega = 1$; (c) $\Omega = 2$.

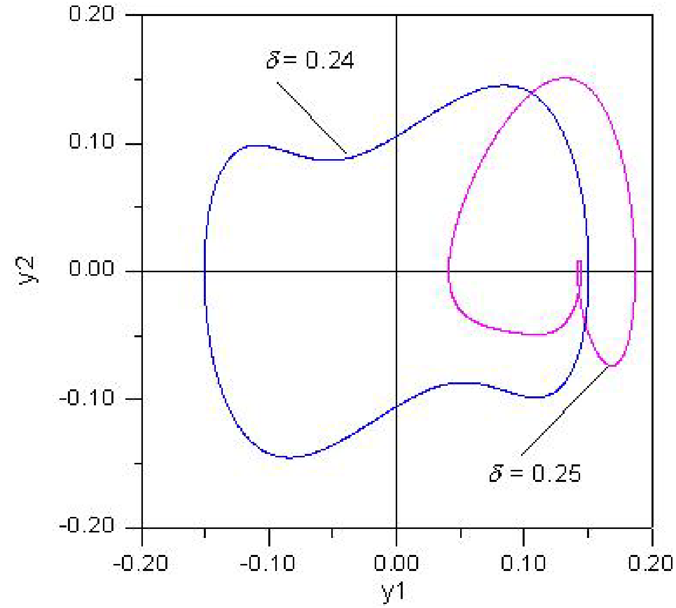


Figure 3. Steady state response for $\Omega = 1$ and two forcing amplitudes $\delta = 0.24$ and $\delta = 0.25$.

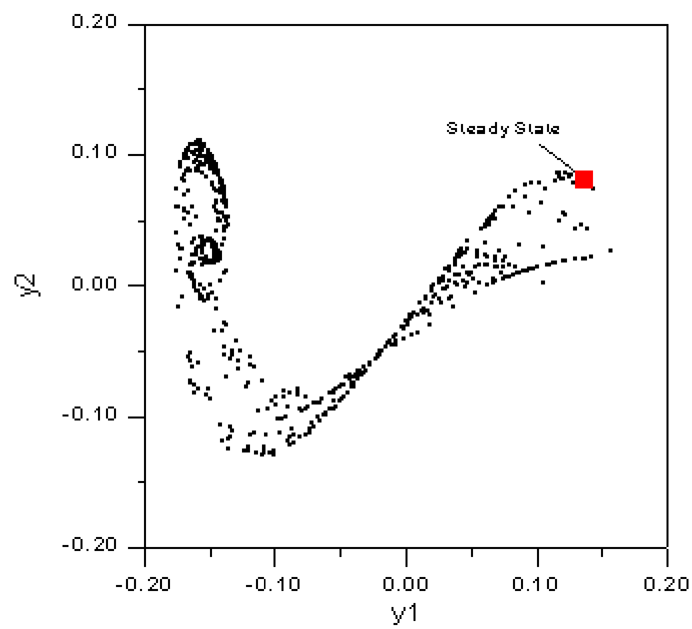
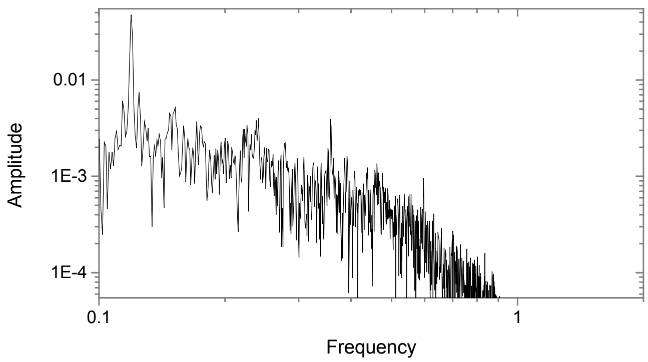
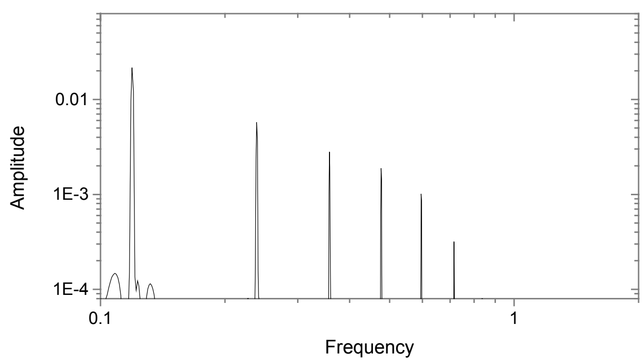


Figure 4. Transient Strange attractor (Ω = 0.75, $\delta = 0.3$).



(a)



(b)

Figure 5. FFT analysis for Ω = 0.75, δ = 0.3. (a) Transient chaos (500th to 632nd cycles); (b) Steady State (1300th to 1432nd cycles).

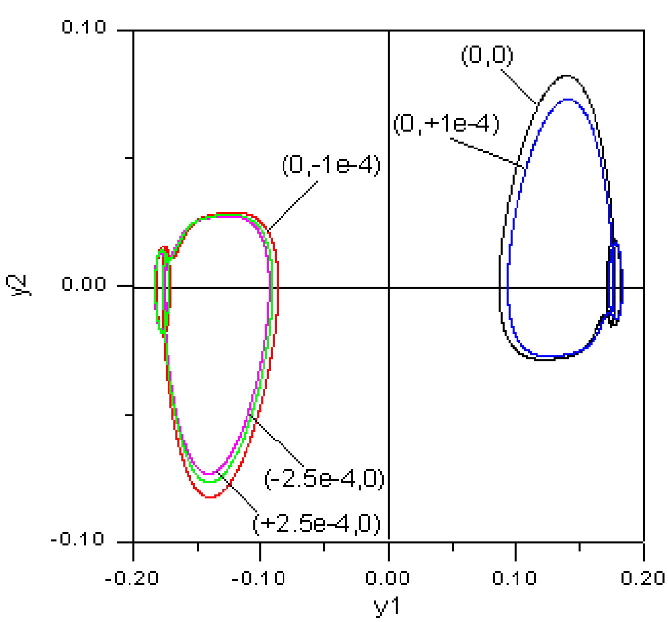


Figure 6. Steady state response for Ω = 0.75, δ = 0.3. Different initial conditions are considered for (y_1, y_2): (0,0), (0,-1e-4), (0,+1e-4), (-2.5e-4,0) and (+2.5e-4,0).

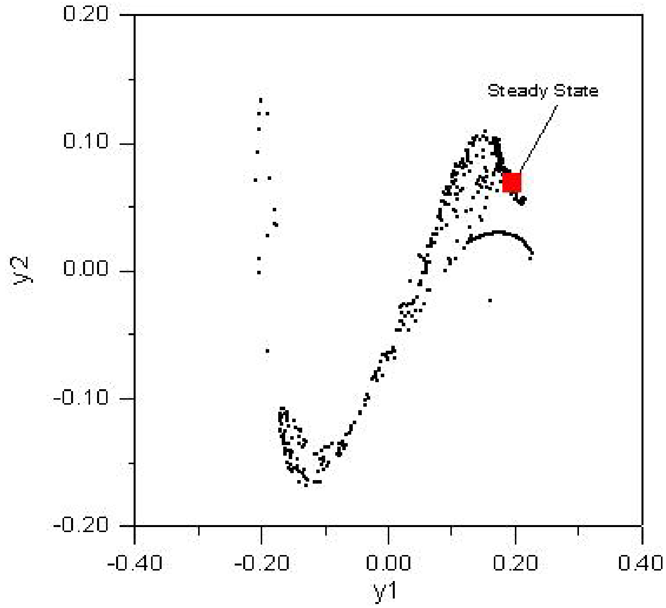
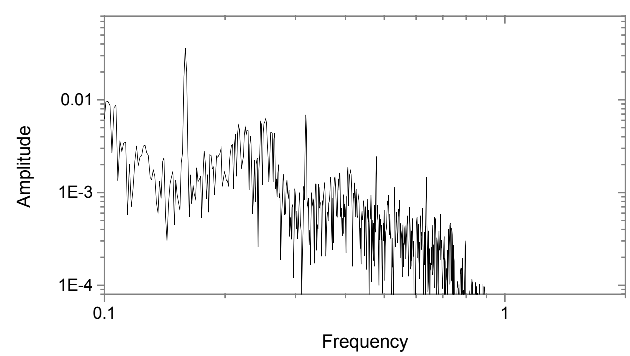
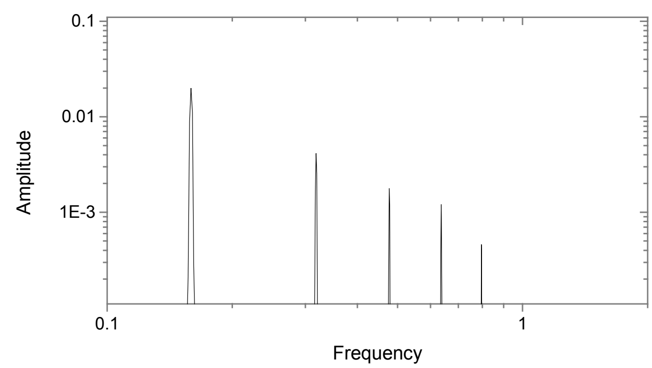


Figure 7. Transient Strange attractor (Ω = 1, δ = 0.5).



(a)



(b)

Figure 8. FFT analysis for Ω = 1, δ = 0.5. (a) Transient chaos (200th to 332nd cycles); (b) Steady state (550th to 682nd cycles).

Conclusions

A dynamic analysis of the Symonds' model for a pin-ended elasto-plastic beam is considered. Kinematic and isotropic hardening are both included on the constitutive model. The numerical method proposed in this work permits the use of a combination of classical algorithms to evaluate the response of elasto-plastic dynamical systems. Numerical investigations show that the beam response presents jump phenomenon which implies that small changes in force amplitudes cause great variations in steady state responses. Transient chaos may also exist in the beam response. High sensitivity on initial conditions may occur in spite of the steady state motion being periodic. These effects may cause practical problems in predicting the response of the beam.

Acknowledgements

The authors would like to acknowledge the support of the Brazilian Research Council (CNPq) and of the Research Foundation of Rio de Janeiro (FAPERJ).

References

Grebogi, C., Ott, E. & Yorke, J.A., 1983, "Crises, Sudden Changes in Chaotic Attractors, and Transient Chaos", *Physisca 7D*, pp.181-200.

Lemaitre, J. & Chaboche, J.L., 1990, "*Mechanics of Solid Materials*", Cambridge University Press.

Moon, F.C., 1992, "*Chaotic and Fractal Dynamics*", John Wiley.

Mullin, T., 1993, "*The Nature of Chaos*", Clarendon Press.

Ortiz, M., Pinsky, P.M. & Taylor, R.L., 1983, "Operator Split Methods for the Numerical Solution of the Elastoplastic Dynamic Problem", *Computer Methods in Applied Mechanical Engineering*, Vol. 39, pp. 137-157.

Poddar, B., Moon, F.C. & Mukherjee, S., 1988, "Chaotic Motion of an Elastic-Plastic Beam", *ASME Journal of Applied Mechanics*, Vol. 55, pp.185-189.

Pratap, R., Mukherjee, S. & Moon, F.C., 1994, "Dynamic Behavior of a Bilinear Hysteretic Elasto-Plastic Oscillator, Part I: Free Oscillations", *Journal of Sound and Vibration*, Vol. 172(3), pp. 321-337.

Pratap, R., Mukherjee, S. & Moon, F.C., 1994, "Dynamic Behavior of a Bilinear Hysteretic Elasto-Plastic Oscillator, Part II: Oscillations Under Periodic Impulse Forcing", *Journal of Sound and Vibration*, Vol. 172(3), pp. 339-358.

Savi, M.A. & Pacheco, P.M.C.L., 1997, "Non-Linear Dynamics of an Elasto-Plastic Oscillator with Kinematic and Isotropic Hardening", *Journal of Sound and Vibration*, Vol. 207(2), pp.207-226.

Shanley, F.R., 1947, "Inelastic Column Theory", *Journal of the Aeronautical Sciences*, Vol. 14, pp. 261-267.

Simo, J.C. & Taylor, R.L., 1985, "Consistent Tangent Operators for Rate-Independent Elastoplasticity", *Computer Methods in Applied Mechanical Engineering*, Vol. 48, pp. 101-118.

Symonds, P.S. & Yu, T.X., 1985, "Counterintuitive Behavior in a Problem of Elastic-Plastic Beam Dynamics", *ASME Journal of Applied Mechanics*, Vol. 52, pp. 517-522.

