Composites Science and Technology 238 (2023) 110041

Contents lists available at ScienceDirect

Composites Science and Technology

journal homepage: www.elsevier.com/locate/compscitech

Theoretical justification of Tsai's modulus based on micromechanical analysis

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ARTICLE INFO

Keywords:
Micromechanics of composites
Tsai's modulus
Effective elastic properties
Asymptotic homogenization
VSPKc model

ABSTRACT

This paper deals with the application of micromechanics for the determination of lamina effective properties. Micromechanical models can be considered as the first step of the composite material mechanical design and the traditional estimative of the effective elastic properties presents the drawback of uncertainties related to the fibers' properties and matrix curing effect. On this basis, Tsai's modulus was proposed based on average normalized properties for carbon fiber reinforced polymers (CFRP). On one hand, this approach has the advantage to be a simple procedure with good results; on the other hand, its generality needs to be confirmed because it is based on empirical approach. The present work develops a theoretical justification of the Tsai's modulus by comparing it with two micromechanical models: asymptotic homogenization, a rigorous analytical model; and VSPKc, a model based on mechanical principles of the traditional rule of mixture. Ranges of average properties for polymeric matrices and carbon fibers are selected, generating 65,536 data with the micromechanical models to be employed for each one of the five independent lamina's elastic properties. Results show that the difference between Tsai's modulus and the average normalized properties estimated by micromechanical models is smaller than 15% for all the properties. In this regard, the investigation provides a theoretical justification of Tsai's modulus approach.

1. Introduction

Composite design is associated with intrinsic multiscale characteristics [1]. Experimental tests are related to high cost and time, which imposes limits to their use [2]. On the other hand, numerical methods based on geometric discretization, like the finite element method, are popular and suitable [3,4] being usually associated with a prohibitive computational cost for multiscale optimization procedures [5]. In this regard, the use of analytical formulations is essential approach to be applied [5].

A strategy for multiscale analysis is to develop an analytical microscale model and perform a macroscale numerical analysis [6]. The contributions from the World-Wide Failure Exercise (WWFE) to macromechanics must be highlighted, where several failure criteria were compared against blinded experimental data [7,8]. An alternative discussion about failure of composite materials can be found in Ref. [9]. Additionally, some recent advances have been obtained for macromechanical stress analysis using Lekhnitskii formalism [10], Stroh formalism [11] and Carrera unified formulation [12], among others.

Regarding micromechanics [13], presented a general overview of micromechanical modeling. Besides, the authors have been carrying out an effort to compare analytical micromechanical models with experimental data available in the literature for elastic properties [14–16] and strengths [17,18]. A large set of data have been used, allowing a proper analysis of the best strategies.

Concerning the elastic properties [14], evaluated a set of 188 experimental data and compared with 10 micromechanical analytical models. The authors highlighted that the asymptotic homogenization model employing a symmetric square unit cell obtained the best estimation without the need of calibrated parameters. Since the asymptotic

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https://doi.org/10.1016/j.compscitech.2023.110041
Received 15 February 2023; Received in revised form 5 April 2023; Accepted 10 April 2023
Available online 18 April 2023

homogenization model obtained closer estimation for square unit cell than for hexagonal unit cell [14,16] developed a model based on the rule of mixture considering a symmetric square unit cell. This novel model, namely VSPKc, also considered element associations in series and in parallel for transversal and shear load, and the unit cell is split into elements to enrich the geometry description. VSPKc model obtained the best estimations considering the rule of mixture-based models when compared with experimental data without the need of calibrated parameters. These conclusions point to the use of the asymptotic homogenization and VSPKc model as good references for the estimation of elastic properties.

[2] claimed that the trace of stiffness matrix can be assumed as a material property for a CFRP lamina, defining normalized material properties with constant values. On this basis, just one experimental test is required to characterize the lamina elastic properties. This methodology has becoming popular in the literature due to its simplicity and good estimations [19–21]. Usually, empirical formulations are useful due to simplicity, allowing approximate estimations for complex problems. For instance, the well-known Tsai-Wu failure criterion was published in 1971 [22] with an empirical proposal to estimate the term that represents the interaction between normal stress components on the material damage, becoming popular and being used for several purposes. Recently, some efforts have been developed to obtain a theoretical justification of this term, which certainly enlarges the model reliability [23,24].

The present paper deals with a theoretical justification of the Tsai's modulus. The strategy is to establish a comparison of lamina effective properties predicted by different micromechanical models. In this regard, asymptotic homogenization and VSPKc models are employed as references since they have presented good agreement with experimental data [14,16].

Since carbon fibers are transversally isotropic, the micromechanical models require 8 inputs from constituents: fiber volume fraction, V_f; fiber longitudinal elastic modulus, E_1^f; fiber transversal elastic modulus, E_2^f; fiber in-plane shear modulus, G_{12}^f; fiber out-of-plane shear modulus, G_{23}^f; fiber in-plane Poisson's ratio, ν_{12}^f; matrix elastic modulus, E^m; and matrix Poisson's ratio, ν^m. The output of these models are lamina's effective elastic properties: longitudinal elastic modulus, E_1; transversal elastic modulus, E_2; in-plane shear modulus, G_{12}; out-of-plane shear modulus, G_{23}; and in-plane Poisson's ratio, ν_{12}. On the other hand, Tsai's modulus requires just one lamina's property, E_1, to compute the other 4 lamina properties using the normalized trace relation, E_2, G_{12}, G_{23} and ν_{12}. Note that the trace relation is independent of V_f.

Although it is expected that the micromechanical models are able to obtain closer estimation than Tsai's modulus [15], showed that the errors from Tsai's modulus are close to those obtained using the asymptotic homogenization model when compared with the same set of experimental data. The main drawbacks of the micromechanical models are related to the reliability of the constituents' properties used as input and fiber random distribution. Considering the fibers, the accuracy of experimentally measured properties is questionable due to the fiber small diameters (around 10 μm), introducing errors for the estimation of the effective properties. Additionally, matrix properties can vary due to manufacture process and ambient conditions [25]. investigated the influence of fiber random distribution introducing the concept of degree of nonuniformity. Tsai's modulus treats these issues using average normalized properties and proposing a simple set of equations.

This paper evaluates, compares, and identifies similarities among asymptotic homogenization, VSPKc model and Tsai's modulus. Although only two micromechanical models are of concern, other models present the same trends. Therefore, the outcome of this paper is to show that Tsai's modulus is consistent with micromechanical models, presenting close estimation compared with experimental data. In this regard, it should be pointed out that these approaches were compared with experimental data in the following references: asymptotic homogenization [14]; Tsai's modulus [15]; and VSPKc model [16]. For the best authors knowledge, only [26] treated this issue, considering Halpin-Tsai model to evaluate two CFRP laminae, indicating a good agreement. The present investigation aims to expand this comparison, considering a larger range of carbon fiber and polymeric matrix properties, as well as improving the micromechanical modeling. On this basis, a theoretical justification of the Tsai's modulus is achieved.

After this Introduction, Tsai's modulus is presented in Section 2 and the micromechanical models are overviewed in Section 3. Results and discussions about the model estimations, as well as the constituents' properties range, are presented in Section 4. The main conclusions are summarized in Section 5.

2. Tsai's modulus

Tsai's modulus is based on experimental evidence that establishes that the trace of the stiffness matrix is a material property, being defined as follows [2,15,27–31],

$$tr(\mathbf{C}) = \frac{E_1}{E_1^*} = \frac{E_2}{E_2^*} = \frac{G_{12}}{G_{12}^*} = \frac{G_{23}}{G_{23}^*} \tag{1}$$

$$\nu_{12}^* = \nu_{12} \tag{2}$$

where $\mathbf{C}$ is the stiffness matrix, $tr(\mathbf{C})$ is trace of stiffness matrix and the upper index "*" denotes the normalized properties.

Considering that the effective properties can be normalized by the trace of the stiffness matrix, the authors proposed a normalized stiffness matrix, $\mathbf{C}^*$, based on a set of experimental properties for different CFRP laminae. For typical CFRP, $\mathbf{C}^*$ is given by [19]

$$\mathbf{C}^* = \frac{\mathbf{C}}{tr(\mathbf{C})} = \begin{bmatrix} 0.752 & 0.0268 & 0.0268 & 0 & 0 & 0 \\ 0.0268 & 0.0568 & 0.0272 & 0 & 0 & 0 \\ 0.0268 & 0.0272 & 0.0568 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0148 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0261 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0261 \end{bmatrix} \tag{3}$$

Hence, the normalized compliance matrix can be computed by $\mathbf{S}^* = (\mathbf{C}^*)^{-1}$ and the lamina normalized properties are

$$E_1^* = \frac{1}{S_{11}^*} = 0.7349 \tag{4}$$

$$\nu_{12}^* = -E_1^* S_{12}^* = 0.319 \tag{5}$$

$$E_2^* = \frac{1}{S_{22}^*} = 0.04351 \tag{6}$$

$$G_{23}^* = \frac{1}{S_{44}^*} = 0.0148 \tag{7}$$

$$G_{12}^* = \frac{1}{S_{66}^*} = 0.02610 \tag{8}$$

With the trace-normalized properties, known as Tsai's modulus, the influence of fiber volume fraction, temperature, moisture absorption and residual stresses from curing can be neglected and just one macromechanical elastic property must be measured.

3. Micromechanical models

This section deals with a summary of micromechanical models, specifically the asymptotic homogenization and the VSPKc model. Although both models consider a symmetric square unit cell, their formulations are distinct. The asymptotic homogenization is based on the theory of elasticity using power series. On the other hand, the VSPKc is based on the theory of elasticity, considering mechanical association of elements is series and in parallel, together with the classical rule of mixture.

3.1. Asymptotic homogenization

Asymptotic homogenization is a rigorous modeling approach that allows the determination of effective elastic properties of composite materials [5,32,33]. Concerning the effective elastic properties of unidirectional laminae with symmetric square unit cell, the analytical solution for isotropic fibers was developed by Ref. [32] and later generalized for transversally isotropic fiber by Ref. [34].

The asymptotic homogenization modeling considers a two-scale approach with macroscopic, x_i, and microscopic variables, $y_i = x_i/\varepsilon$, where $\varepsilon \to 0$ is a small parameter. Note that $\partial/\partial x_i \to [(\partial/\partial x_i) + (1/\varepsilon)(\partial/\partial y_i)]$. The displacement field is assumed to be an asymptotical power series of ε as follows

$$u_i^{(\varepsilon)} = u_i^{(0)}(\mathbf{x}) + \varepsilon u_i^{(1)}(\mathbf{x},\mathbf{y}) + \dots \tag{9}$$

Since the displacements are written as power series of ε, the strain and stress components can also be defined as power series ε using the strain-displacement and constitutive equations,

$$e_{ij}^{(\varepsilon)} = e_{ij}^{(0)}(\mathbf{x},\mathbf{y}) + \varepsilon e_{ij}^{(1)}(\mathbf{x},\mathbf{y}) + \dots \tag{10}$$

$$\sigma_{ij}^{(\varepsilon)} = \sigma_{ij}^{(0)}(\mathbf{x},\mathbf{y}) + \varepsilon \sigma_{ij}^{(1)}(\mathbf{x},\mathbf{y}) + \dots \tag{11}$$

where the following relations are obtained evaluating the powers of ε

$$e_{ij}^{(0)}(\mathbf{x},\mathbf{y}) = \frac{1}{2}\left(\frac{\partial u_i^{(0)}}{\partial x_j} + \frac{\partial u_j^{(0)}}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i^{(1)}}{\partial y_j} + \frac{\partial u_j^{(1)}}{\partial y_i}\right) \tag{12}$$

$$e_{ij}^{(1)}(\mathbf{x},\mathbf{y}) = \frac{1}{2}\left(\frac{\partial u_i^{(1)}}{\partial x_j} + \frac{\partial u_j^{(1)}}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i^{(2)}}{\partial y_j} + \frac{\partial u_j^{(2)}}{\partial y_i}\right) \tag{13}$$

$$\sigma_{ij}^{(0)}(\mathbf{x},\mathbf{y}) = c_{ijkl}(\mathbf{y})\frac{\partial u_k^{(0)}}{\partial x_l} + c_{ijkl}(\mathbf{y})\frac{\partial u_k^{(1)}}{\partial x_l} \tag{14}$$

$$\sigma_{ij}^{(1)}(\mathbf{x},\mathbf{y}) = c_{ijkl}(\mathbf{y})\frac{\partial u_k^{(1)}}{\partial x_l} + c_{ijkl}(\mathbf{y})\frac{\partial u_k^{(2)}}{\partial x_l} \tag{15}$$

By considering the equilibrium requirement for the term of $O(\varepsilon^{-1})$ and assuming that $u_i^{(0)} = u_i^{(0)}(\mathbf{x})$, the following displacement relation can be established

$$u_i^{(1)}(\mathbf{x},\mathbf{y}) = N_{ijk}(\mathbf{y})\frac{\partial u_k^{(0)}(\mathbf{x})}{\partial x_l} \tag{16}$$

where $N_{ijk}(\mathbf{y})$ is a third order tensor with the components being Y-periodic functions.

By integrating the terms of $O(\varepsilon^0)$ over the unit cell domain, the effective elastic tensor is defined by

$$\langle c_{ijkl}\rangle = \frac{1}{|Y|}\int\left[c_{ijkl}(\mathbf{y}) + c_{ijmn}(\mathbf{y})\frac{\partial N_{mkl}(\mathbf{y})}{\partial y_n}\right]d\mathbf{y} \tag{17}$$

In summary, the main challenge of the asymptotic homogenization is the determination of the tensor $N_{ijk}(\mathbf{y})$, where it is necessary to solve six independent problems: two anti-plane strain problems and four plane strain problems [32,34].

These solutions are usually obtained using infinite power series to satisfy the boundary conditions. According to Ref. [35], the infinite series may be properly truncated on the second term to converge. The authors proposed the following closed-form equations

$$k = k_f V_f + k_m(1-V_f) - \frac{V_f(k_m-k_f)^2 K}{m_m} \tag{18}$$

$$l = l_f V_f + l_m(1-V_f) - \frac{V_f(k_m-k_f)(l_m-l_f)K}{m_m} \tag{19}$$

$$n = n_f V_f + n_m(1-V_f) - \frac{V_f(k_m-k_f)^2 K}{m_m} \tag{20}$$

$$p = p_m - 2V_f p_m P \tag{21}$$

$$m = m_m - V_f(m_m - m_f)M \tag{22}$$

$$m' = m_m - V_f(m_m - m_f)M' \tag{23}$$

where $k_{m,f} = 0.5(C_{11}^{m,f} + C_{12}^{m,f})$, $m_{m,f} = C_{66}^{m,f}$, $n_{m,f} = C_{33}^{m,f}$, $l_{m,f} = C_{12}^{m,f}$ and $p_{m,f} = C_{44}^{m,f}$ are the constituent properties; $\mathbf{C}^m$ and $\mathbf{C}^f$ are the matrix and fiber elastic tensors defined by

$$\mathbf{C}^m = \begin{bmatrix} \frac{1}{E^m} & -\frac{\nu^m}{E^m} & -\frac{\nu^m}{E^m} & 0 & 0 & 0 \\ -\frac{\nu^m}{E^m} & \frac{1}{E^m} & -\frac{\nu^m}{E^m} & 0 & 0 & 0 \\ -\frac{\nu^m}{E^m} & -\frac{\nu^m}{E^m} & \frac{1}{E^m} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu^m)}{E^m} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2(1+\nu^m)}{E^m} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu^m)}{E^m} \end{bmatrix}^{-1} \tag{24}$$

$$\mathbf{C}^f = \begin{bmatrix} \frac{1}{E_1^f} & -\frac{\nu_{12}^f}{E_1^f} & -\frac{\nu_{12}^f}{E_1^f} & 0 & 0 & 0 \\ -\frac{\nu_{12}^f}{E_1^f} & \frac{1}{E_2^f} & \frac{2G_{23}^f - E_2^f}{2E_2^f G_{23}^f} & 0 & 0 & 0 \\ -\frac{\nu_{12}^f}{E_1^f} & \frac{2G_{23}^f - E_2^f}{2E_2^f G_{23}^f} & \frac{1}{E_2^f} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}^f} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{12}^f} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}^f} \end{bmatrix}^{-1} \tag{25}$$

Additionally, the following parameters are defined based on a truncated infinite series [35],

$$K = C\left\{V_m + \frac{29.7904(1+\kappa_m)CR^8}{B^{-1} + R^6\left[AB^{-1}r + g + 29.7904DR^2\right]}\right\} \tag{26}$$

$$P = \frac{\chi_p}{\left[1 + V_f\chi_p - 29.7904\chi_p{}^2R^8\right]} \tag{27}$$

$$M = \frac{1+\kappa_m}{\left[1+\kappa_m(m_f/m_m)\right](1 + R^2H^- - I)} \tag{28}$$

$$M' = \frac{1+\kappa_m}{\left[1+\kappa_m(m_f/m_m)\right](1 + R^2H^+ - I')} \tag{29}$$

$$\kappa_{f,m} = 1 + 2(m_{f,m}/k_{f,m}) \tag{30}$$

$$\chi_p = \frac{p_m - p_f}{p_m + p_f} \tag{31}$$

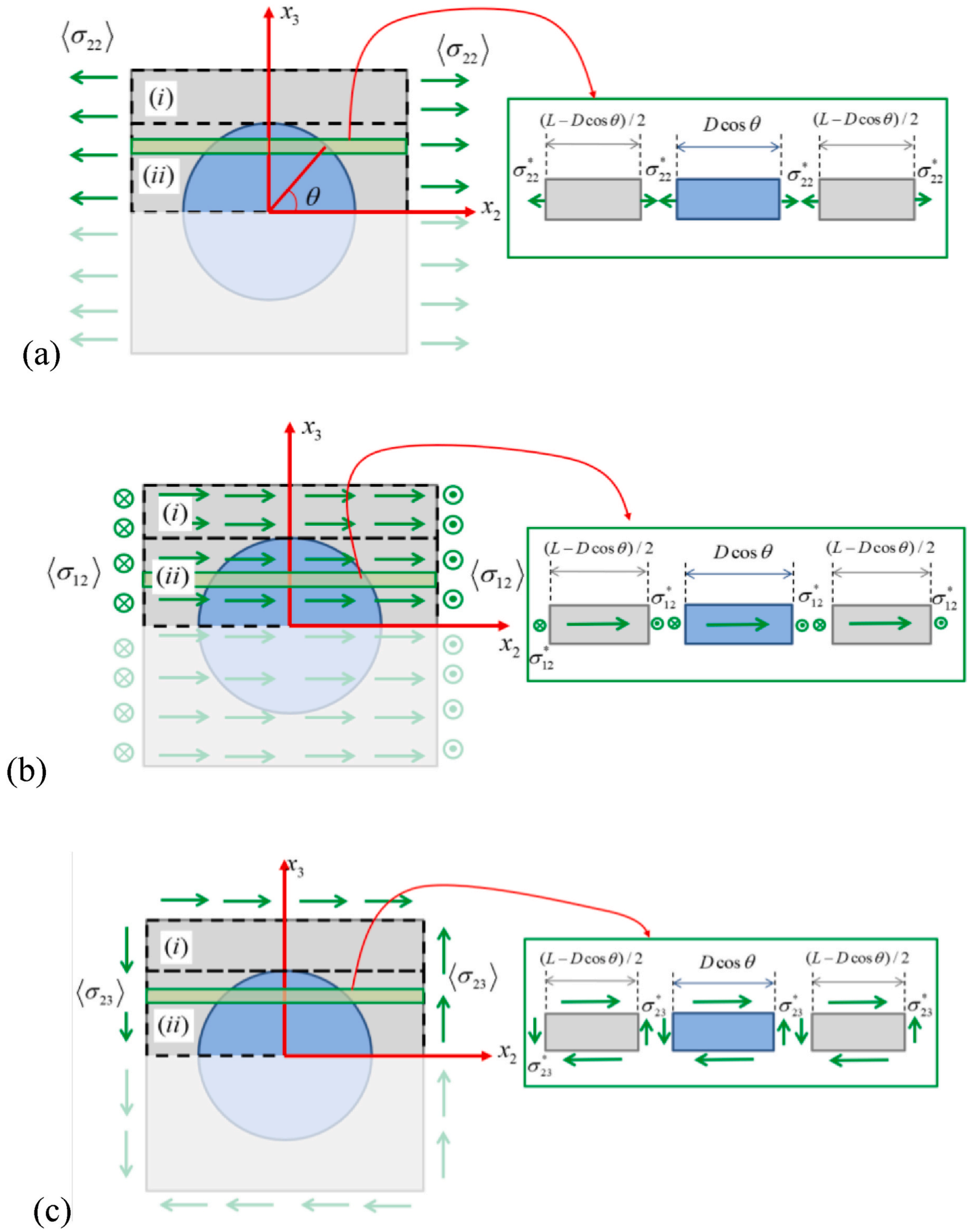

Fig. 1. Unit cell for the VSPKc model and the infenitesimal element equilibrium: (a) transversal load; (b) in-plane shear load; (c) out-of-plane shear.

$$H^+ = Ar_1 + Bk_m\pi + B[5.0153 + g_1] \tag{32}$$

$$H^- = Ar_1 + Bk_m\pi - B[5.0153 + g_1] \tag{33}$$

$$I = \frac{R^{12}(Ar_2 - Bg_2)(Ar_3 - Bg_3)}{1 + R^{10}(Ar_4 - Bg_4)} \tag{34}$$

$$I' = \frac{R^{12}(Ar_2 + Bg_2)(Ar_3 + Bg_3)}{1 + R^{10}(Ar_4 + Bg_4)} \tag{35}$$

$$A = \frac{[\kappa_m(m_f/m_m) - \kappa_f]}{[(m_f/m_m) + \kappa_f]}B \tag{36}$$

$$B = \frac{[1 - (m_f/m_m)]}{[1 + \kappa_m(m_f/m_m)]} \tag{37}$$

$$C = \frac{m_m}{[m_m + k_m V_f + k_f(1 - V_f)]} \tag{38}$$

Table 1
Range of constituents' properties for typical CFRP available in the literature.

	V_f	E^m [GPa]	ν^m	E_1^f [GPa]	E_2^f [GPa]	G_{12}^f [GPa]	G_{23}^f [GPa]	ν_{12}^f
Lower Bound	0.4	3	0.34	220	10	15	5	0.2
Upper Bound	0.7	5	0.4	300	20	30	7	0.3

$$D = 2[(k_2/k_1) - 1]C \tag{39}$$

where $R = \sqrt{V_f/\pi}$, $r = 13312.0294R^{10}$, $r_1 = 29.7904R^6$, $r_2 = 281.6277R^6$, $r_3 = 1408.1385R^6$, $r_4 = 13312.0294R^6$, $g = -893.7124R^2 + 270.9309$, $g_1 = -18.9073R^2$, $g_2 = -119.1616R^2 + 27.0931$, $g_3 = -595.8082R^2 + 135.4655$, and $g_4 = -18197.4824R^2 + 4889.7209$.

Finally, the effective elastic constants computed using the asymptotic homogenization method are given by

$$E_1 = \frac{nk - l^2}{k} \tag{40}$$

$$E_2 = 4m' \left[\frac{nk - l^2}{n(k + m') - l^2}\right] \tag{41}$$

$$\nu_{12} = -\frac{l}{2k} \tag{42}$$

$$G_{12} = p \tag{43}$$

$$G_{23} = m' \tag{44}$$

3.2. VSPKc model

The micromechanical VSPKc model is briefly discussed in this section considering the derivation proposed by Ref. [16]. By considering a square unit cell with length L and fiber's diameter D, the fiber volume fraction is defined by $V_f = \pi D^2/4L^2$. For a longitudinal load, the equations are similar to the classical rule of mixture where fiber and matrix are assumed to be elements in parallel. Hence, the longitudinal elastic modulus and the in-plane Poisson's ratio are computed using the following equations

$$E_1 = E_1^f V_f + E^m(1 - V_f) \tag{45}$$

$$\nu_{12} = \nu_{12}^f V_f + \nu^m(1 - V_f) \tag{46}$$

For transversal and shear loads, the unit cell is split into two parts, as illustrated in Fig. 1: part (i) presents just matrix; and part (ii) presents both matrix and fiber. Parts (i) and (ii) are in parallel and inside part (ii), for any infinitesimal element located in an angle θ, fiber and matrix are in series. Therefore, the equilibrium requirement is defined as follows

$$\langle\sigma_{22}\rangle = \frac{2}{L}\left(\int_{D/2}^{L/2} \sigma_{22}^{(i)} dx_3 + \int_0^{D/2} \sigma_{22}^{(ii)} dx_3\right) \tag{47}$$

In part (i), the integral is solved directly because there is only matrix,

$$\int_{D/2}^{L/2} \sigma_{22}^{(i)} dx_3 = \frac{1}{2} E^m \varepsilon_{22}^{(i)} (L - D) \tag{48}$$

In part (ii), each infinitesimal element has the effective transversal modulus similar to the classical results of the rule of mixture for transversal load

$$E_2^* = E^m \left\{\frac{1}{1 + \left[\left(E^m/E_2^f\right) - 1\right] V_f^*}\right\} \tag{49}$$

where $V_f^* = (D/L)\cos\theta = 2\sqrt{V_f/\pi}\cos\theta$. Note that $E_2^* = E_2^*(\theta)$ since $V_f^* = V_f^*(\theta)$.

Using the transformation $dx_3 = (D/2)\cos\theta d\theta$, the equilibrium integral in part (ii) is

$$\int_0^{D/2} \sigma_{22}^{(ii)} dx_3 = E^m \varepsilon_{22}^{(ii)} \frac{D}{2}\left[\frac{\pi}{2a_{22}} - \frac{\ln\left(a_{22} + \sqrt{a_{22}^2 - 1}\right)}{a_{22}\sqrt{a_{22}^2 - 1}}\right] \tag{50}$$

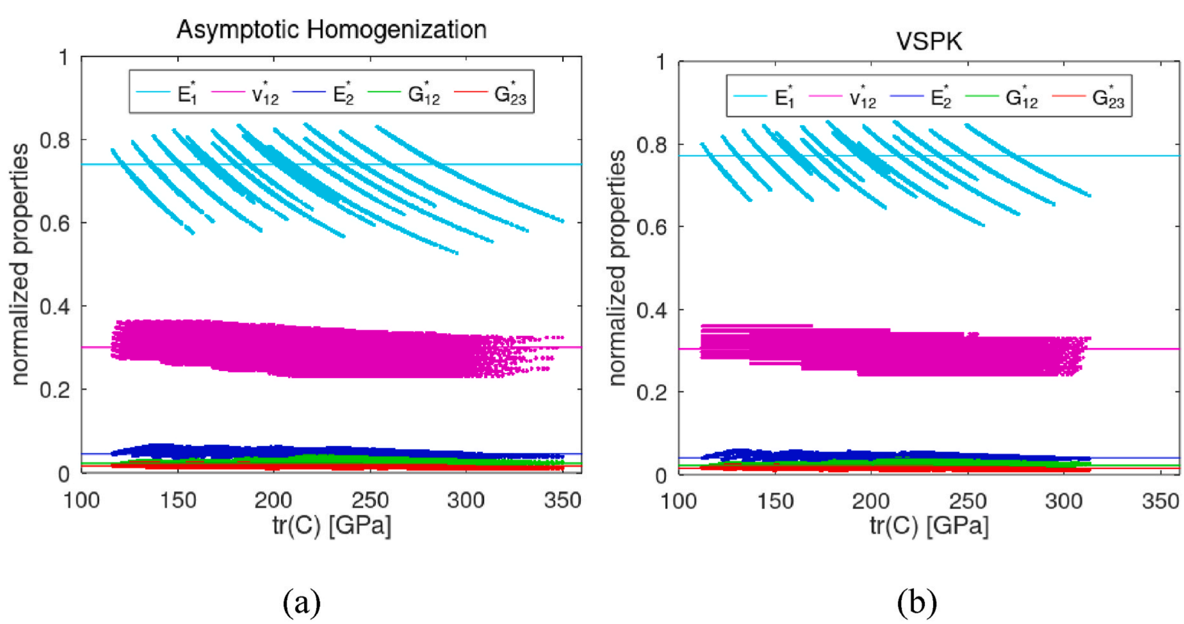

Fig. 2. Average normalized properties using micromechanical models: (a) Asymptotic Homogenization; (b) VSPKc.

Table 2
Statistical analysis of Asymptotic Homogenization and VSPKc model average normalized properties.

Normalized Properties	Asymptotic Homogenization			VSPKc		
	AV	SD	CV [%]	AV	SD	CV [%]
E_1^*	0.7405	4.286×10^{-2}	5.79	0.7721	3.717×10^{-2}	4.81
E_2^*	0.0458	6.639×10^{-3}	14.50	0.0405	6.413×10^{-3}	15.84
G_{12}^*	0.0232	4.558×10^{-3}	19.65	0.0224	4.402×10^{-3}	19.68
G_{23}^*	0.0165	2.236×10^{-3}	13.55	0.0155	2.267×10^{-3}	14.65
ν_{12}^*	0.302	1.018×10^{-2}	9.27	0.304	2.693×10^{-2}	8.86

*AV: Average; SD: Standard Deviation; CV: Coefficient of Variation.

Table 3
Normalized properties computated Asymptotic Homogenization and VSPKc model divided by Tsai's Modulus.

Normalized Properties	Asymptotic Homogenization / Tsai′s Modulus	VSPK / Tsai′s Modulus
E_1^*	1.0076	1.0506
E_2^*	1.0529	0.9310
G_{12}^*	0.8889	0.8582
G_{23}^*	1.1149	1.0473
ν_{12}^*	0.9467	0.9530

where $a_{22} = 2[(E^m/E_2^f) - 1]\sqrt{V_f/\pi}$.

Considering the geometrical compatibility, $\langle\varepsilon_{22}\rangle = \varepsilon_{22}^{(i)} = \varepsilon_{22}^{(ii)}$. Manipulating Eq. 47–50, the effective transversal elastic modulus is

$$E_2 = E^m\left\{1+2\sqrt{\frac{V_f}{\pi}}\left[\frac{\pi}{2a_{22}}-\frac{\ln\left(a_{22}+\sqrt{a_{22}^2-1}\right)}{a_{22}\sqrt{a_{22}^2-1}}-1\right]\right\} \quad (51)$$

A similar procedure can be carried out to obtain the effective in-plane and out-of-plane shear moduli, resulting in the following equation

$$G_{12} = G^m\left\{1+2\sqrt{\frac{V_f}{\pi}}\left[\frac{\pi}{2a_{12}}-\frac{\ln\left(a_{12}+\sqrt{a_{12}^2-1}\right)}{a_{12}\sqrt{a_{12}^2-1}}-1\right]\right\} \quad (52)$$

$$G_{23} = G^m\left\{1+2\sqrt{\frac{V_f}{\pi}}\left[\frac{\pi}{2a_{23}}-\frac{\ln\left(a_{23}+\sqrt{a_{23}^2-1}\right)}{a_{23}\sqrt{a_{23}^2-1}}-1\right]\right\} \quad (53)$$

where $a_{12} = 2[(G^m/G_{12}^f) - 1]\sqrt{V_f/\pi}$ and $a_{23} = 2[(G^m/G_{23}^f) - 1]\sqrt{V_f/\pi}$.

4. Results and discussion

This section establishes comparison among the presented models, Tsai's modulus, asymptotic homogenization and VSPKc, using experimental data as reference. Table 1 shows the ranges considered in this investigation considering the average properties of carbon fibers and polymeric matrix properties available in the literature [36–40]. The range of each input listed in Table 1 is divided in 4 equally spaced values (for instance, $V_f = [0.4 \quad 0.5 \quad 0.6 \quad 0.7]$), resulting in a set of $4^8 = 65536$ combinations.

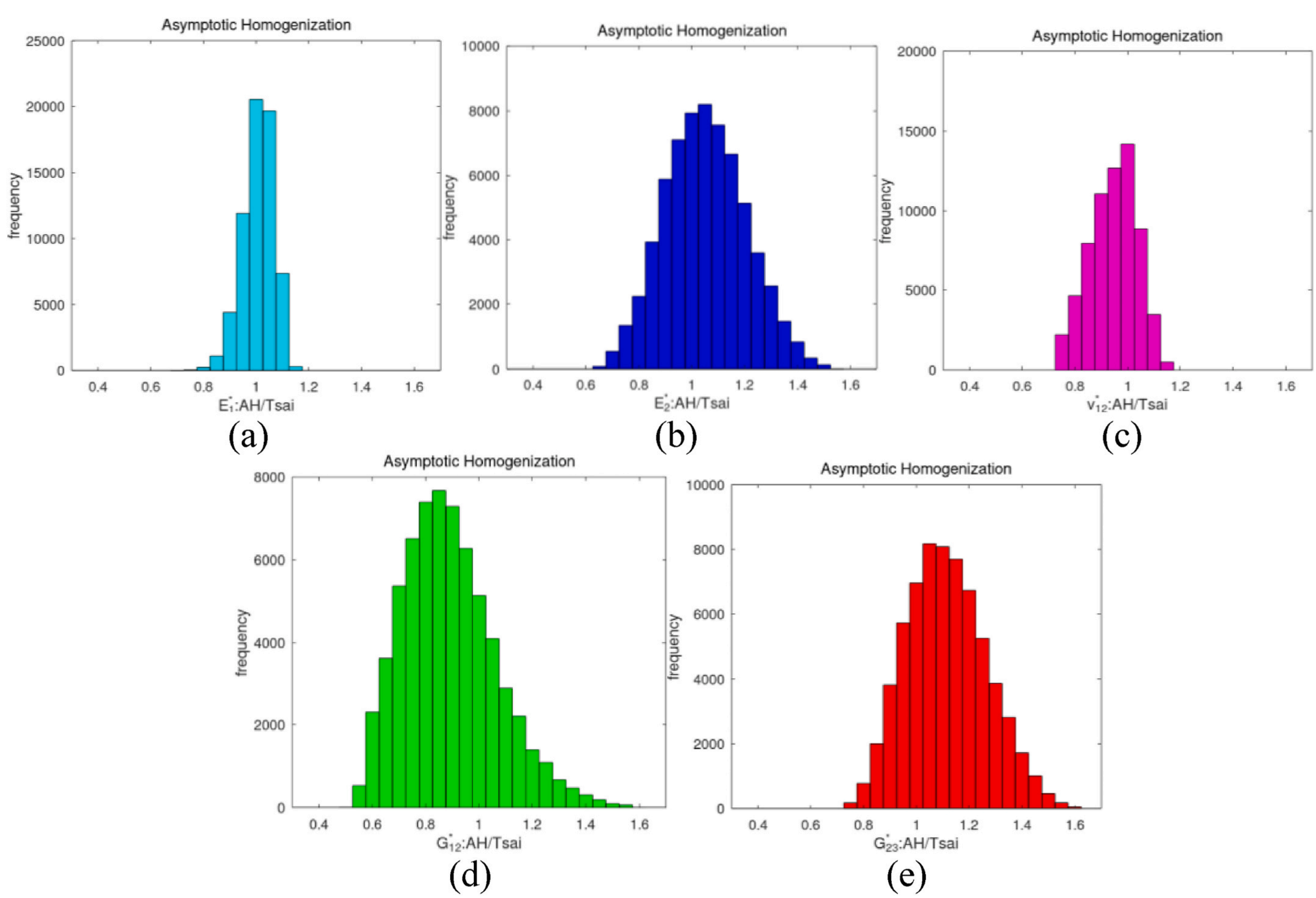

Fig. 3. Histogram of the normalized properties according to the ratio between asymptotic homogenization and Tsai's modulus: (a) E_1^*; (b) E_2^*; (c) ν_{12}^*; (d) G_{12}^*; (e) G_{23}^*.

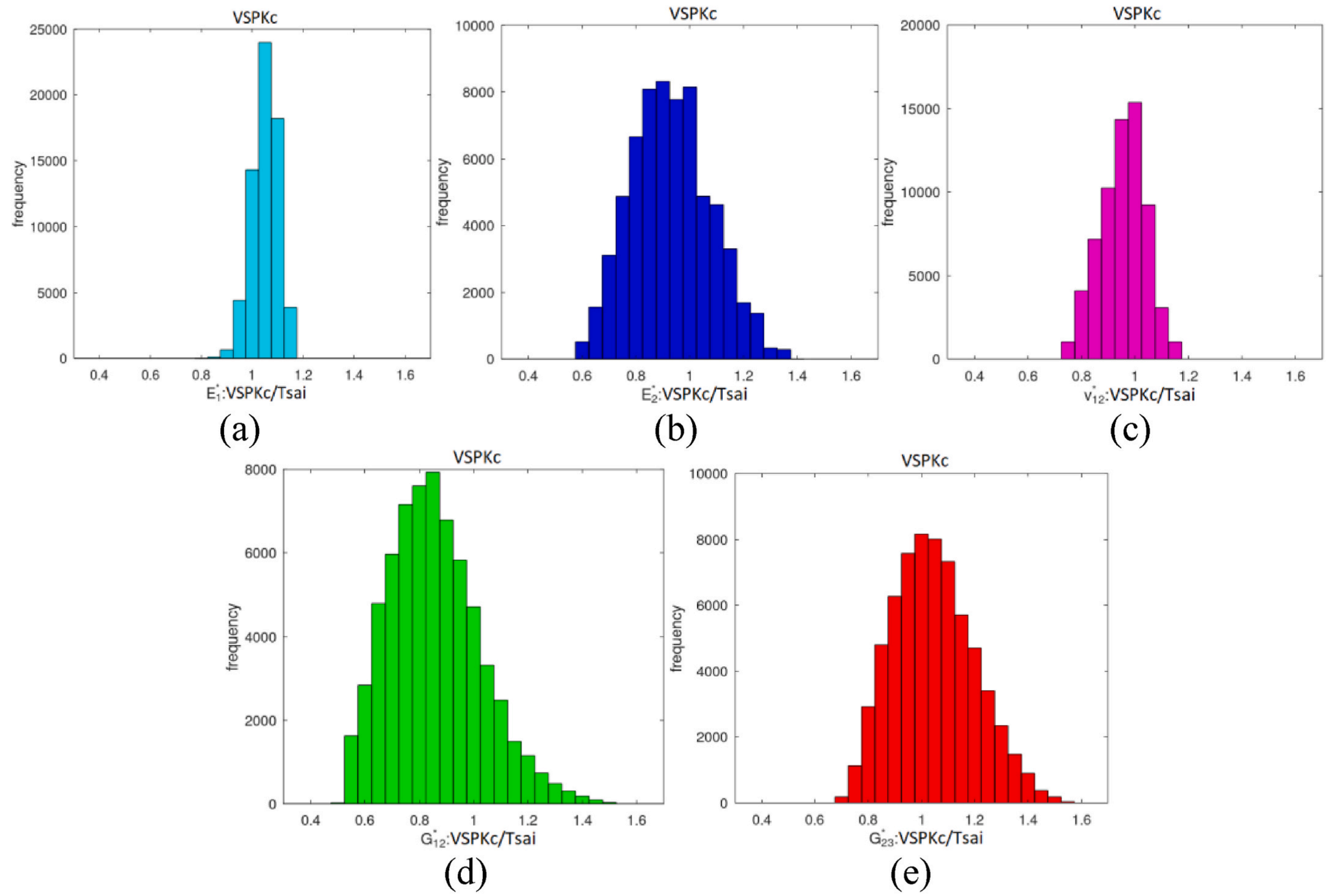

Fig. 4. Histogram of the normalized properties according to the ratio between VSPKc and Tsai's modulus: (a) E_1^*; (b) E_2^*; (c) ν_{12}^*; (d) G_{12}^*; (e) G_{23}^*.

Fig. 2 presents normalized laminae properties computed by micromechanical models, where the dots are the generated data from ranges of constituents' properties and the lines are the average values. In other words, for each normalized property, the combination of V_f, E^m, ν^m, E_1^f, E_2^f, G_{12}^f, G_{23}^f and ν_{12}^f is represented by one dot, while the average of all this dots is represented by the line. Since there is a dispersion of the generated data, the average values, standard deviations and coeffecients of variation are listed in Table 2. The upper limit value of the trace from asymptotic homogenization is slightly higher than from VSPKc. The largest difference from the average normalized properties is for E_2^*, where the value computed by asymptotic homogenization is 13% larger than the value from VSPKc. For all the other normalized properties, the differences between these micromechanical model estimations are smaller than 6%. It is important to highlight that the normalized properties can be assumed insensitive to the V_f with a good precision, they are properties of CFRP.

The ratio between the micromechanical estimations and Tsai's modulus are presented in Table 3. The micromechanical models have similar average values comparing with Tsai's modulus. The largest difference is for G_{12}^*, being smaller than 15%. It is interesting to be pointed out that unidirectional laminae present nonlinear behavior when subjected to in-plane shear loads [38]. A summary of difficulties related to experimental measurements for in-plane shear can be found in Ref. [41]. Hence, the experimental values of G_{12} also tends to increase the uncertainties and to decrease its reliability.

In order to establish a comparison between the micromechanical model and the Tsai modulus, a normalization with Tsai's modulus is performed allowing a direct comparison. Fig. 3 shows the histogram of the normalized properties of associated with asymptotic homogenization while Fig. 4 shows the histograms of the normalized properties related to the VSPKc model. The following conclusions should be highlighted.

i) a normal shape can be realized for all properties;
ii) regarding property E_1^*, the center of the distribution is 1 for asymptotic homogenization and it is slightly moved to rigth for VSPKc;
iii) regarding property E_2^*, the center of the distribution is 1.05 for asymptotic homogenization and 0.9 for VSPKc;
iv) regarding property G_{12}^*, both micromechanical models show the center of the distribution shifted to the left, indicating that the Tsai's modulus estimates larger values than the micromechanical models;
v) regarding property G_{23}^*, the micromechanical models and Tsai's models present a good agreement and the center of the distribution is 1;
vi) regarding property ν_{12}^*, the histograms present a similar behavior when compared with the distributions of E_1^*, where the center is around 1 with a small dispersion.

5. Conclusions

The present paper proposes a theoretical justification for the empirical Tsai's modulus, establishing a comparision with the asymptotic homogenization and VSPKc micromechanical models. The trace-normalized properties are compared with those predicted by analytical micromechanical models. It is shown that the difference among all properties is smaller than 15%, proving a good agreement. The biggest difference is for normalized in-plane shear modulus, G_{12}^{*}, which can be related to the nonlinear behavior of unidirectional laminae subjected to in-plane shear loads. Therefore, the trace approach seems to be an acceptable approach to simplify the design of composite materials. In addition, the average normalized properties from asymptotic homogenization and VSPKc models are discussed, showing a trend for normal distribution.

Author statement

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

Acknowledgements

The authors would like to acknowledge the support of the Brazilian Research Agencies CNPq, CAPES, FAPERJ and the Natural Sciences and Engineering Research Council of Canada (NSERC).

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