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Estimating Attractor Dimension on the Nonlinear Pendulum Time Series

Luiz Fernando P. Franca
Marcelo A. Savi
Instituto Militar de Engenharia
Department of Mechanical and Materials Engineering
22290-270 Rio de Janeiro, RJ. Brazil
savi@ime.eb.br

Chaotic dynamical systems exhibit trajectories in their phase space that converges to a strange attractor. The strangeness of the chaotic attractor is associated with its dimension in which instance it is described by a noninteger dimension. This contribution presents an overview of the main definitions of dimension discussing their evaluation from time series employing the correlation and the generalized dimension. The investigation is applied to the nonlinear pendulum where signals are generated by numerical integration of the mathematical model, selecting a single variable of the system as a time series. In order to simulate experimental data sets, a random noise is introduced in the time series. State space reconstruction and the determination of attractor dimensions are carried out regarding periodic and chaotic signals. Results obtained from time series analyses are compared with a reference value obtained from the analysis of mathematical model, estimating noise sensitivity. This procedure allows one to identify the best techniques to be applied in the analysis of experimental data.
***Keywords**: Chaos, state space reconstruction, attractor dimension, fractals, time series*

Introduction

Fractals have been observed in nature in different situations varying from geometry to physical sciences. Basically, it is possible to categorize fractals into two different groups: solid objects and strange attractors. The first type includes physical objects that exist in ordinary physical space. On the other hand, the second type considers conceptual objects that exist in the state space of chaotic dynamical systems (Theiler, 1990). Mandelbrot (1982) establishes the existence of the *geometry of nature* in contrast with the classical geometry, which provides just a first approximation to the structures of physical objects. Therefore, fractal geometry may be considered as an extension of classical geometry.

The attractor dimension counts the effective number of degrees of freedom in the dynamical system. Chaotic dynamical systems exhibit trajectories in their phase space that converges to a strange attractor. The strangeness of the chaotic attractor is associated with its dimension in which instance it is described by a noninteger dimension. Hausdorff (1919) gave a rigorous definition of dimension that is a basic property of an attractor.

There are a variety of different forms to define or quantify the dimension of an attractor. Farmer *et al*. (1983) presents an overview of these definitions, considering two general types: those that depend only on metric properties and those that depend on the frequency with which a typical trajectory visits different regions of the attractor. Furthermore, there is the Kaplan-Yorke conjecture that defines the Lyapunov dimension calculated from Lyapunov exponents (Kaplan & Yorke, 1983; Wolf *et al*., 1985).

This contribution presents an overview of the main definitions of dimension, discussing their evaluation from time series. Two different algorithms are employed with this aim: the correlation dimension that is determined employing the algorithm proposed by Hegger *et al.* (1999), and the generalized dimension which is evaluated employing the algorithm proposed by Sarraille & Myers (1994). Furthermore, the Lyapunov dimension is estimated from the mathematical model employing the Kaplan & Yorke conjecture, defining a reference value. Since noise contamination is unavoidable in cases of data acquisition, the determination of attractor dimension needs to have no noise sensitivity. Many studies are devoted to evaluate noise suppression and its effects in the analysis of chaos, however, there are a small number of reports devoted to the effects of the system noise on chaos (Ogata *et al*., 1997). The present investigation is applied to the nonlinear pendulum where signals are generated by numerical integration of the mathematical model, selecting a single variable of the system as a time series. In order to simulate experimental data sets, a random noise is introduced in the time series. State space reconstruction and the determination attractor dimensions are carried out regarding periodic and chaotic signals. The number of data points is chosen as the minimum required for a correct estimation of the desirable measure. Results obtained from time series analyses are compared with the reference value estimating noise sensitivity. This procedure allows one to identify the best techniques to be applied in the analysis of experimental data.

Attractor Dimension

A geometrically intuitive notion of dimension, D, is as an exponent that expresses the scaling of an object's bulk with its size: $Bulk \sim Size^D$. Here, *Bulk* may correspond to a volume, a mass, or even a measure of information content, while *Size* is a linear distance. Therefore, the definition of dimension is usually cast as an equation of the form (Theiler, 1990),

$$D = \lim_{Size \to 0} \frac{\log(Bulk)}{\log(Size)}, \qquad (1)$$

where the limit of small size is taken to ensure invariance over coordinate changes. This also implies that dimension is a local quantity and that any global definition of dimension require some kind of averaging.

Different definitions of these quantities imply different measures of dimensions. *Hausdorff* and *capacity* dimensions are some examples of fractal dimensions while *pointwise*, *information* and *correlation* dimensions are examples of dimension of the natural measure. Other definitions may be found in Farmer *et al.* (1983), Mayer-Kress (1985), Paladin & Vulpiani (1987) and Theiler (1990).

Capacity Dimension

Kolmogorov originally defines the capacity of a set which may be considered as the most basic definition of fractal dimension. It may be derived from the notion of counting the number of boxes β, N_β, of size ε, needed to cover the orbit in phase space. Basically, the number of boxes depends on the subspace of the orbit: $N_\beta(\varepsilon) \approx \varepsilon^{-D}$. Therefore, the following definition of the capacity dimension, D_K, may be done,

$$D_K = \lim_{\varepsilon \to 0} \frac{-\log[N_\beta(\varepsilon)]}{\log(\varepsilon)}. \qquad (2)$$

The capacity dimension may be conceived as a simplification of the Hausdorff measure, D_H, which numerical implementation is very difficult. Farmer *et al.* (1983) shows that $D_K \geq D_H$, but for most fractal sets of interest, these dimensions are equal.

Moon (1992) presents two criticisms of the use of capacity dimension as a measure of attractor dimension. The first one is theoretical, because it does not account for the frequency with which the orbit might visit the covering box. The other criticism is computational, because the process of counting the boxes is very time consuming. Furthermore, Liebovitch & Toth (1989) point that the procedure requires too large data points.

Useful algorithms to implement the box-counting procedure are described by Grassberger (1983), Hunt & Sullivan (1986), Giorgilli *et al.* (1986) and Theiler (1987). Further, Liebovitch & Toth (1989) suggest a new algorithm, which is less dependent on data points.

Pointwise Dimension

The pointwise dimension, D_P, is a local measure of the dimension of the fractal set at a point of the attractor. Let a phase space trajectory where a box of side ε, $\beta(\varepsilon)$, centered at the point x_i, is considered. This box contains some points of the trajectory, N_β. Defining the pointwise mass function as the probability of finding a point in this box divided by the total number of points in the orbit, N,

$$\Gamma_i(\varepsilon, x_i) \equiv \frac{N_{\beta_{xi}}(\varepsilon)}{N}, \qquad (3)$$

this allows one to define the pointwise dimension as follows,

$$\hat{D}_P(x_i) = \lim_{\varepsilon \to 0} \frac{-\log \Gamma_i(\varepsilon, x_i)}{\log \varepsilon} . \tag{4}$$

This definition is a local measure, but it is possible to obtain an averaged pointwise definition randomly choosing a set of points $M < N$ and calculating the dimensions at each point. Hence, the following dimension is defined (Moon, 1992).

$$D_P = \frac{1}{M} \sum_{i=1}^{M} \hat{D}_P(x_i) . \tag{5}$$

A useful algorithm to implement the average pointwise dimension is described by Mayer-Kess (1987).

Information Dimension

The information dimension, D_I, is a generalization of the capacity dimension measure, which takes into account the frequency with which the trajectories visits each covering box. Since the dimension may be understood as something that counts the degrees of freedom of the system, it is possible to establish a definition where the evaluation of how many bits of information are necessary to specify a point to a given accuracy is of interest. In general, $I(\varepsilon) = -D \log_2(\varepsilon)$ bits of information are needed to specify the position of a unity D-dimensional box to an accuracy ε. Shannon's formula is employed in order to evaluate the needed average information to define one box (Theiler, 1990):

$$I(\varepsilon) = -\sum_{i=1}^{N} \Gamma_i(\varepsilon) \log \Gamma_i(\varepsilon) , \tag{6}$$

where Γ_i is the probability measure of the ith box. This relation leads to an expression for the information dimension

$$D_I = \lim_{\varepsilon \to 0} \frac{-I(\varepsilon)}{\log(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{\sum \Gamma_i(\varepsilon) \log \Gamma_i(\varepsilon)}{\log(\varepsilon)} . \tag{7}$$

The information is a measure of unpredictability in a system. In order to establish a relation with the capacity measure, one considers that the probability Γ_i are equal for all boxes, and hence $I(\varepsilon) = \log N_{\beta(\varepsilon)}$, meaning that $D_I = D_K$. In general, it can be shown that $I(\varepsilon) < \log N_{\beta(\varepsilon)}$, and hence $D_I \leq D_K$ (Farmer *et al.*, 1983; Moon, 1992). Hegger *et al.* (1999), Grassberger *et al.* (1989), Badii & Politi (1985) and Theiler (1988) describe different approaches and algorithms for the information dimension.

Correlation Dimension

The correlation dimension, D_C, is another probabilistic dimension, which represents one of the most popular forms to measure the attractor dimension. This measure has been successfully used by many experimentalists (Moon, 1992) and is defined as follows,

$$D_C = \lim_{\varepsilon \to 0} \frac{-\log \sum_i \Gamma_i^2(\varepsilon)}{\log \varepsilon} , \tag{8}$$

where Γ_i is a correlation function of two points. Therefore, this dimension measures the probability to find a pair of random points in an elementary box. Notice that this measure is different from the information dimension, which considers the probability to find just one point in a given box. Hence, one tries to count the number of pair distances.

Grassberger & Proccacia (1983) and Takens (1983) suggest the use of the correlation integral, $C(\varepsilon,N)$, to estimate $\sum_i \Gamma_i^2$. This integral represents a direct arithmetic average of the pointwise mass function (Theiler, 1990). For a signal with N points, the correlation integral $C(\varepsilon,N)$ has a dynamic range of $O(N^2)$, which is twice the range which appears in the box-counting algorithms.

The popularity of the correlation algorithm is based on its straightforward implementation. Grassberger & Procaccia (1983) and Hegger *et al.* (1999) describe useful algorithms to estimate this dimension.

Generalized Dimension

The generalized dimension was introduced by Hentschel & Procaccia (1983) and independently by Grassberger (1983). The notion of generalized dimension first arose out of a need to understand why various algorithms gave different answers for dimension. A further motivation came from the need to characterize more fully fractals with nonuniform measure. Therefore, the generalized dimension is defined as follows,

$$D_G = \lim_{\varepsilon \to 0} \frac{-I_G(\varepsilon)}{\log \varepsilon}, \tag{9}$$

where,

$$I_G = \frac{1}{1-G} \log \sum_{i=1}^{N} [\Gamma_i(\varepsilon)]^G . \tag{10}$$

With this definition, the measure of dimension is related to the value assumed for G. When $G = 0$, $I_G = I_0$, and D_G corresponds to the capacity dimension. If $G = 1$, it is possible to consider $G = 1 + \gamma$, with $\gamma \to 0$, to define an expression related to the information dimension. When $G = 2$, the definition represents the correlation dimension.

Lyapunov Dimension

The Lyapunov dimension, D_L, associated with the Kaplan-Yorke conjecture (Kaplan & Yorke, 1983), is calculated from the Lyapunov exponents and takes into account the dynamical properties of the attractor. In order to establish a definition of this measure, consider the spectrum of Lyapunov exponents, in a decreasing order. Therefore, the following definition is regarded (Frederickson *et al.*, 1983)

$$D_L = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|}, \tag{11}$$

where j is defined from the conditions,

$$\sum_{i=1}^{j} \lambda_i > 0 \text{ and } \sum_{i=1}^{j+1} \lambda_i < 0 . \tag{12}$$

Moon (1992) shows that Lyapunov and information dimensions are equivalent, *i.e.*, $D_I = D_L$.

State Space Reconstruction

Consider the motion of a nonlinear pendulum where θ defines its position, α is the linear viscous damper parameter and ω_n is associated with the natural frequency of the system. Moreover, a harmonic forcing with amplitude ρ and frequency Ω are concerned. With these assumptions, the dynamical system is governed by the well-known equation of motion

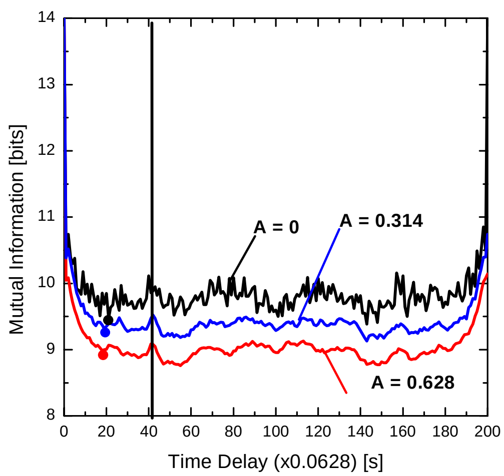
$$\ddot{\theta} + \alpha\dot{\theta} + \omega_n^2 \sin(\theta) = \rho \cos(\Omega t). \quad (13)$$

This equation may be rewritten as a system, $\dot{u} = f(u,t), u \in R^3$, where $u_1 = x = \theta$, $u_2 = y = \dot{\theta}$ and $u_3 = \Omega t$. Numerical simulations are carried out employing the fourth-order Runge-Kutta method with time steps less than $\Delta t = 2\pi / 100\Omega$. For all simulations, parameters $\alpha = 0.2$ and $\omega_n = \Omega = 1.0$ are considered. In order to simulate experimental data sets, a signal $s = x + \eta$ is defined where $\eta = AR(-1,+1)$ depicts noise, with A being the amplitude, and $R(-1,+1)$ represents random number within the interval $(-1,+1)$. If $\eta = 0$, there is no noise and an ideal experimental data is simulated. In this article, two other noise levels are contemplated: $A = 0.314$ and $A = 0.628$, representing, respectively, 5% and 10% of the maximum signal amplitude. Basically, periodic and chaotic signals are considered. The number of data points, N, is chosen as the minimum required for a correct estimation of the desirable measure.

The first problem on signal analysis is to convert the time series into state vectors, which is done, using state space reconstruction. The basic idea of this reconstruction is that a signal contains information about unobserved state variables, which can be used to predict the present state. Therefore, a scalar time series, $s(t)$, may be used to construct a vector time series that is equivalent to the original dynamics from a topological point of view. The state space reconstruction needs to form a coordinate system to capture the structure of orbits in state space. The method of delay coordinates could be done using lagged variables, $s(t+\tau)$, where τ is the time delay. Then, considering an experimental signal, $s(t)$, where $t = t_0 + (n-1)\Delta t$ with $n = 1, 2, 3,\ldots, N$, it is possible to use a collection of time delays to create a vector in a D_e-dimensional space, $u(t)$, which represents the reconstructed dynamics of the system (Ruelle, 1979; Packard *et al.*, 1980; Takens, 1981).

$$u(t) = \{s(t), s(t+\tau), \ldots, s(t+(D_e-1)\tau)\}^T. \quad (14)$$

This contribution employs the average mutual information method to determine time delay (Fraser & Swinney, 1986) and the method of false nearest neighbors to estimate embedding dimension (Rhodes & Morari, 1997).



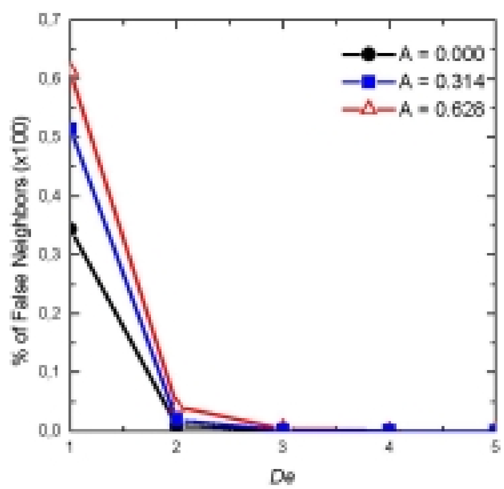


Figure 1. Period-2 motion.

(a) Mutual information versus time delay; (b) Percentage of false neighbors versus embedding dimension.

In order to analyze the state space reconstruction of the nonlinear pendulum employing the method of delay coordinates, a period-2 signal with N = 20,000 is considered. Figure 1 presents results of the mutual information and the false neighbors analysis, for different noise levels. Concerning the time delay determination, there is a difficulty to determine the first minimum of the information curve when A = 0, ideal signal. Nevertheless, time delay may be estimated defining a region limited by the first global maximum of the curve I versus τ (vertical line). Under this assumption, the time delay can be chosen as the first global minimum of this region furnishing values that present good results. The analysis for different noise levels furnishes the following values: τ = 1.319s when A = 0; τ = 1.256s when A = 0.314; τ = 1.256s when A = 0.628. On the other hand, embedding dimension analysis points to D_e = 3, which is in agreement with the mathematical model. It should be noted that noise does not have any influence in results.

Following the determination of delay parameters, the method of delay coordinates can be applied in order to reconstruct the state space. The numerical state space of the motion is presented in Figure 2 together with the reconstructed spaces for different noise levels. The comparison among the reconstructed state spaces with the numerical allows one to observe just a small coordinate change from one to another.

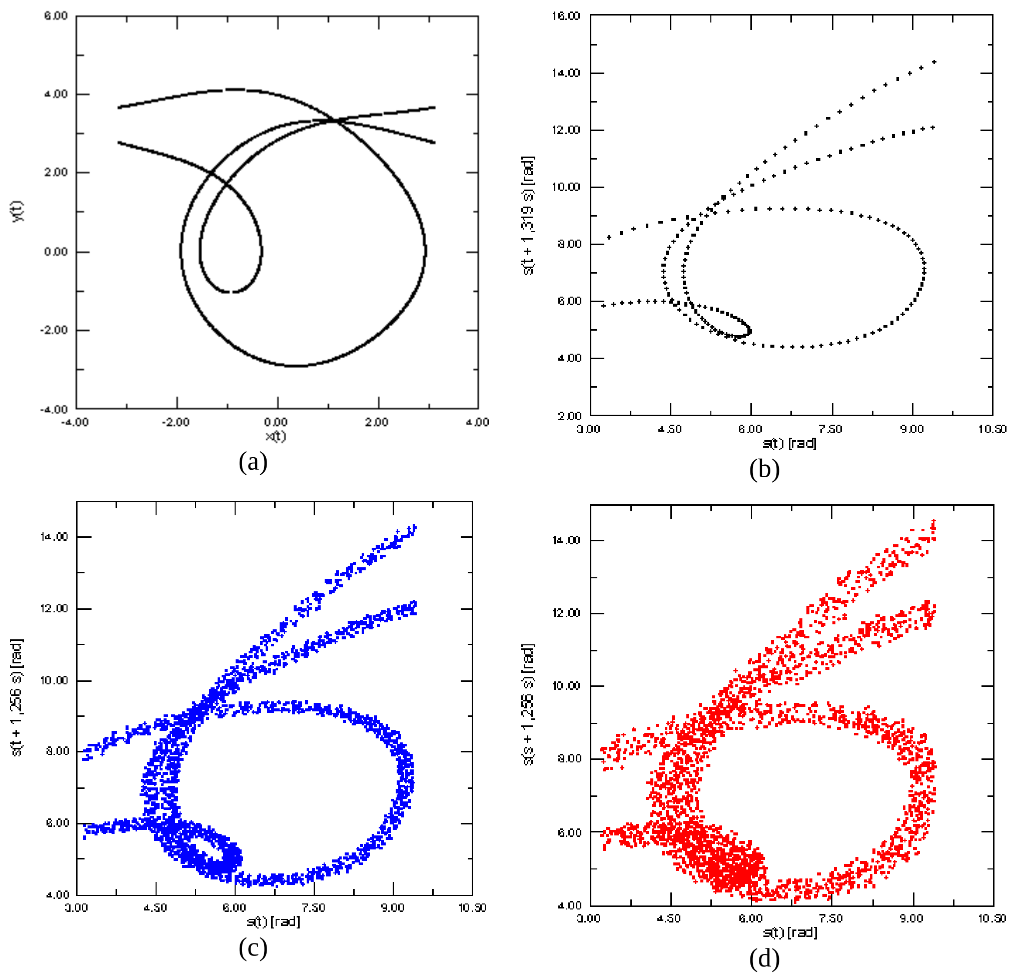
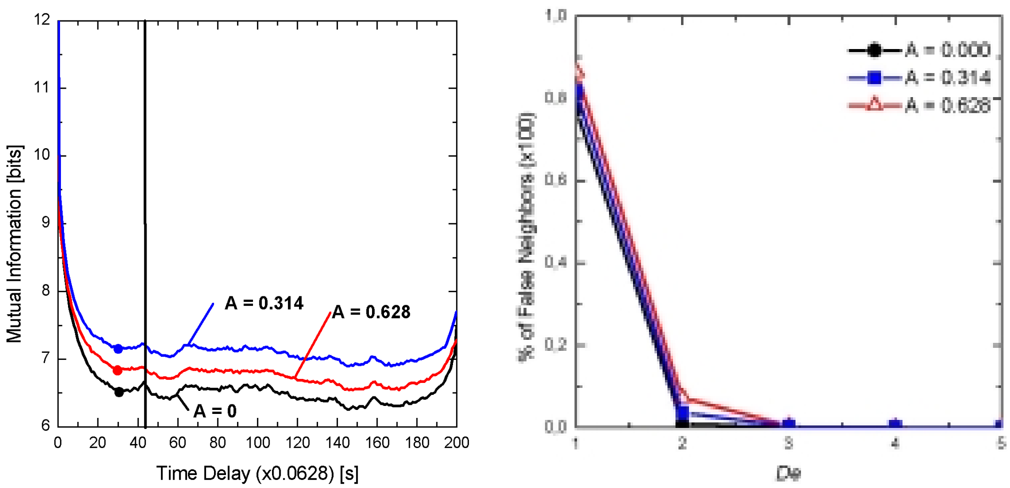


Figure 2. Phase space for period-2 motion.
(a) Numerical; (b) Reconstructed, A = 0; (c) Reconstructed, A = 0.314; (d) Reconstructed, A = 0.628.

The forthcoming analysis regards to a chaotic signal with N = 20,000. Figure 3 considers results of the mutual information and the false neighbors analysis, for different noise levels. The same procedure applied in the determination of time delay of periodic signal can be employed here, resulting on the following values: τ = 2.262s when A = 0; τ = 1.885s when A = 0.314; τ = 1.822s when A = 0.628. Once again, the analysis of the embedding dimension points to D_e = 3 and the noise does not have significantly influence in results.

Figure 4 presents the Poincaré map obtained either by numerical simulation or by reconstruction using the method of delay coordinates for three noise levels: A = 0, A = 0.314 and A = 0.628. A strange attractor is clearly identified where there is a fractal-like structure. Once again, the comparison among reconstructed state spaces with the numerical allows one to observe just a small coordinate change from one to another.



(a) (b)

Figure 3. Chaotic motion.

(a)Mutual information versus time delay; (b) Percentage of false Neighbors versus embedding dimension $\dot{\theta}(t)$.

Nonlinear Pendulum Attractor Dimension

In this section, the estimation of the nonlinear pendulum attractor dimension is considered employing two different algorithms. The correlation dimension is determined employing the algorithm proposed by Hegger *et al.* (1999) while the generalized dimension is evaluated employing the algorithm proposed by Sarraille & Myers (1994). Furthermore, the Lyapunov dimension is estimated from the mathematical model employing the Kaplan & Yorke conjecture, defining a reference value.

At first, correlation dimension of a period-2 signal is conceived. Figure 6 shows the correlation dimension for an ideal signal (A = 0) with N = 1,000, and different values of embedding dimension. The slope of the linear range in Figure 6a is related to the position of the horizontal range in Figure 6b and represents the value obtained for the correlation dimension. In this signal, the value associated with the linear (or horizontal) range is not easy to be obtained which introduces difficulties to evaluate the correlation dimension. When noisy periodic data sets are analyzed, the measure is easier to be evaluated when compared with the previous case. Figure 7-8 show results for noise signals with N = 8,000 and two different noise levels, A = 0.314 and A = 0.628, respectively. Employing linear regression, results show that the correlation dimension is between 0.86 and 1.32, while the reference value obtained from numerical simulation is 1.00. Notice that this range not only includes non-integer values but also is sensitive to the embedding dimension. Therefore, it is not possible to identify a periodic motion since it is difficulty to predict the attractor dimension.

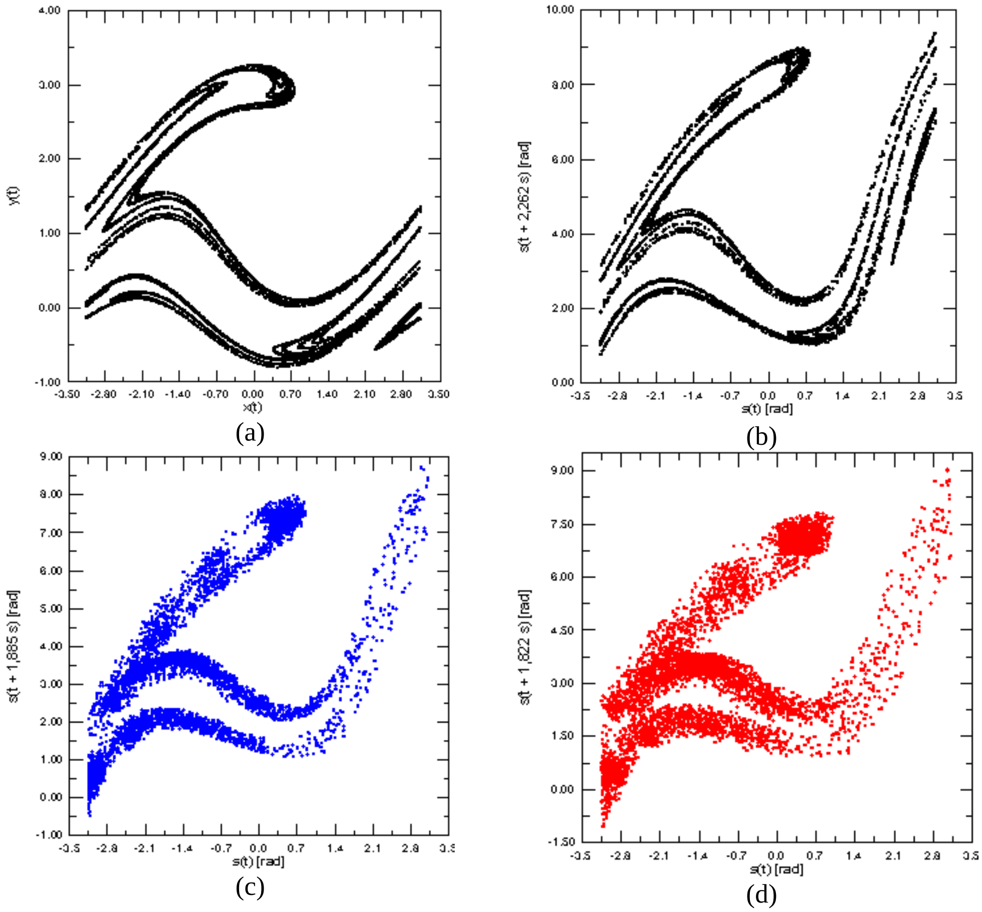


Figure 4. Strange attractors.

(a) Numerical; (b) Reconstructed, A = 0; (c) Reconstructed, A = 0.314, (d) Reconstructed, *A* = 0.628.

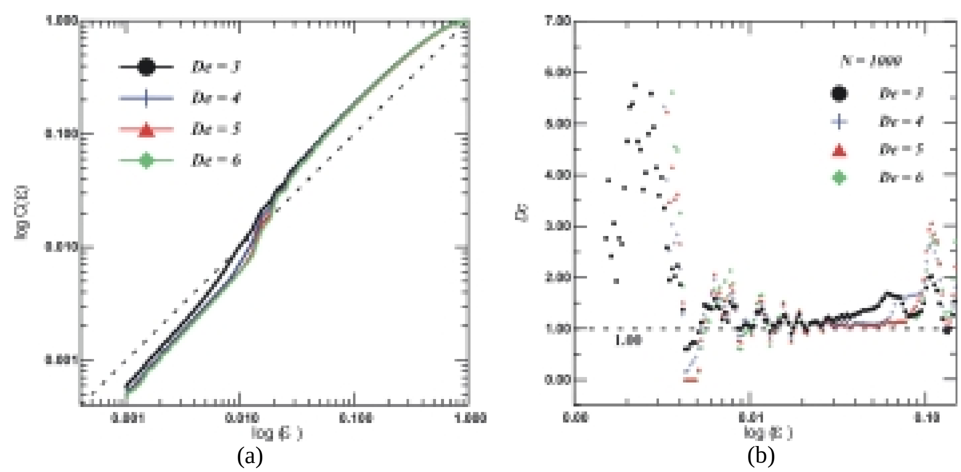
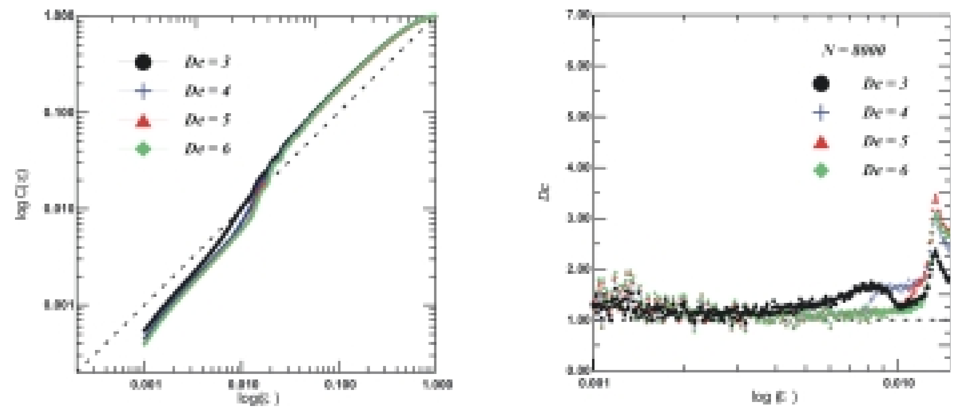
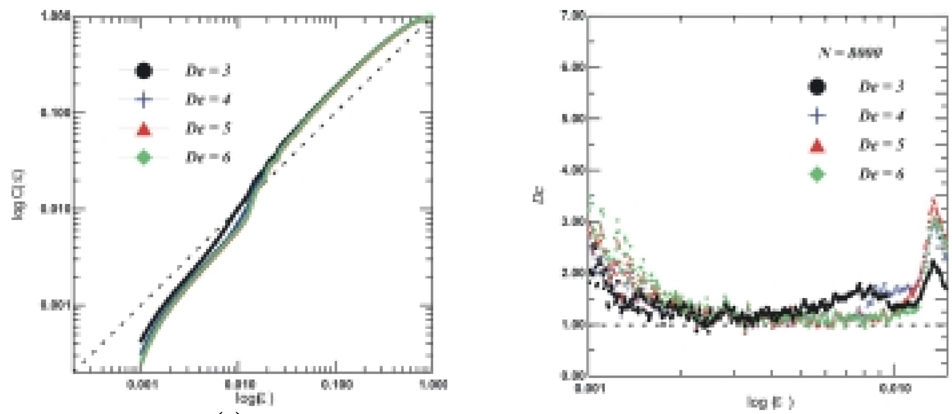


Figure 5. Correlation dimension of a periodic signal (*A* = 0).
(a) log C(ε) versus log(ε); (b)D_C versus log(ε).



(a) (b)

Figure 6. Correlation dimension of a periodic signal (*A* = 0.314).
(a) log C(ε) versus log(ε), (b)D_C versus log(ε)

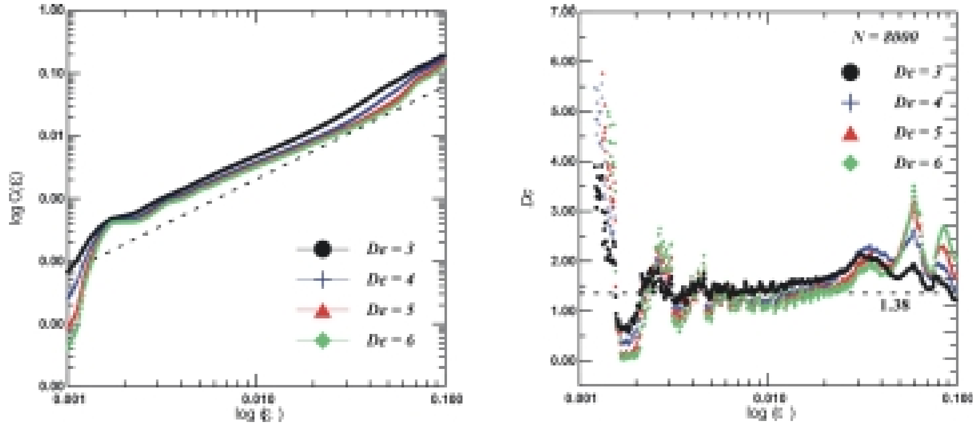


(a) (b)

Figure 7. Correlation dimension of a periodic signal (A = 0.618).
(a) log C(ε) versus log(ε); (b) D_C versus log(ε).

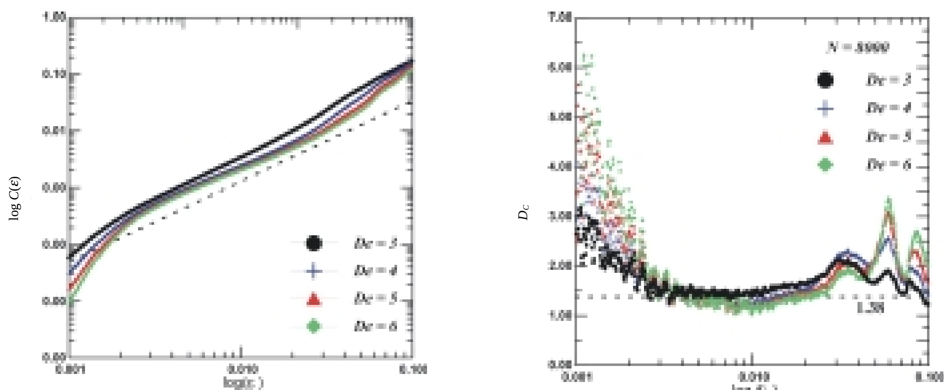
A chaotic signal with *N* = 8,000 is now focused. The correlation dimension for ideal chaotic signal (*A* = 0) and different values of embedding dimension is presented in Figure 9. Employing linear regression, results show that the dimension value is between 1.00 and 1.50, while the reference value obtained from numerical simulation is 1.38. Once again, notice that the alteration of embedding dimension has a great influence on results.

At this point, noise chaotic data sets with *N* = 8,000 are considered. Figure 10 shows the correlation dimension for *A* = 0.314 while Figure 11 conceives *A* = 0.628. Notice that even though the measure of the dimension is similar to the one obtained for the ideal signal, the noise reduces the range where the curve is linear or horizontal.



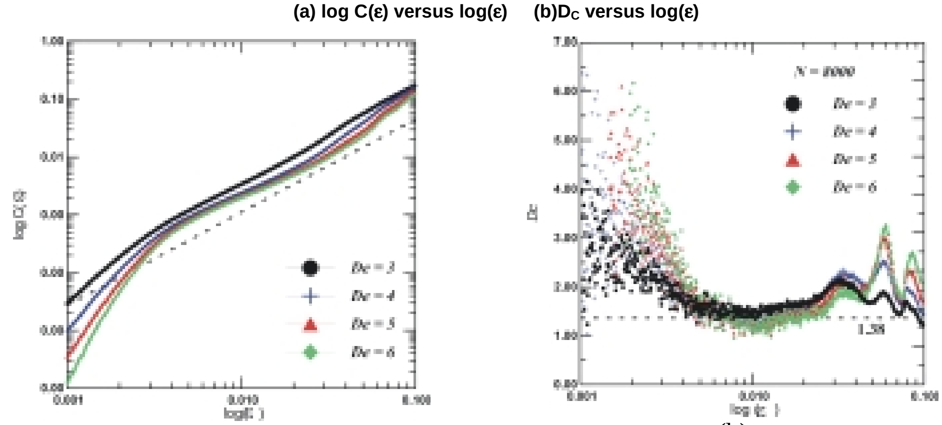
(a) (b)

Figure 8. Correlation dimension of a chaotic signal (A = 0).
(a) log C(ε) versus log(ε); (b)D_C versus log(ε).



(a) (b)

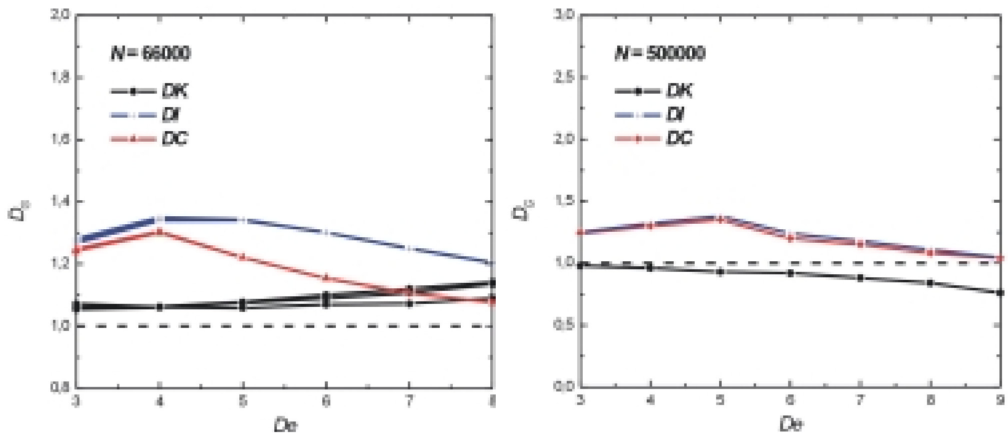
Figure 9. Correlation dimension of a chaotic signal (A = 0.314).
(a) log C(ε) versus log(ε) (b)D_C versus log(ε)



(a) (b)

Figure 10. Correlation dimension of a chaotic signal (A = 0.628).
(a) log C(ε) versus log(ε); (b)D_C versus log(ε).

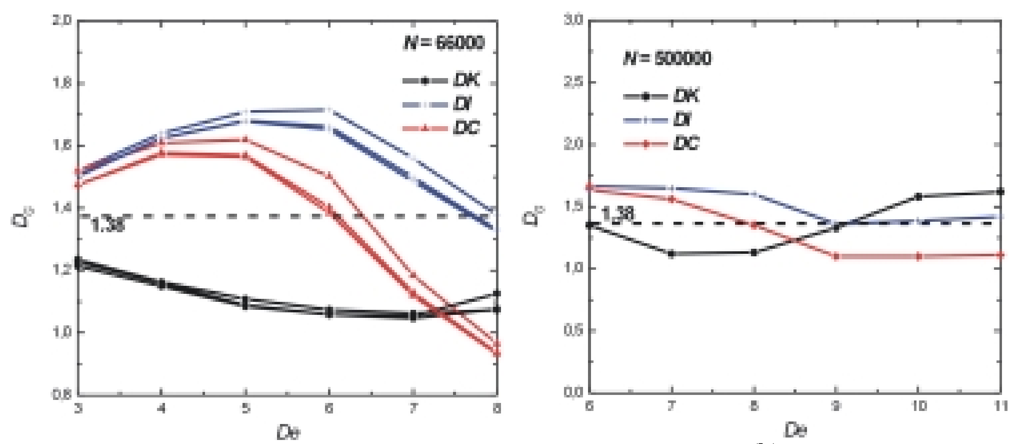
The forthcoming analysis focuses on the generalized dimension. In order to start the analysis, a period-2 signal is considered. The number of data points is important to obtain a good approximation of this measure and the noise does not affect results significantly (Figure 12a). By considering great values of embedding dimension (Figure 12b), the dimension measure converges to values which are near to the reference value (1.0). Nevertheless, it should be emphasized that this procedure needs large data points, which is unfeasible to be applied to experimental data.



(a) (b)

Figure 11. Generalized dimension of a periodic signal.
(a) *N* = 66000; (b) *N* = 500000.

The generalized dimension of a chaotic signal is now in order. The noise does not influence results in a significantly form while the embedding dimension is critical (Figure 13a). This implies that it is necessary a signal with large data points to obtain a correct estimation of the generalized measure. Regarding great values of embedding dimension (Figure 13b), the dimension measure converges to values which are near to the reference value (1.38) but, once again, this procedure is unfeasible to be applied to experimental data.



(a) (b)

Figure 12. Generalized dimension of a chaotic signal.
(a) *N* = 66000; (b) *N* = 500000.

Conclusions

This contribution presents an overview of the main techniques employed on the estimation of attractor dimension from time series. The main purpose is to evaluate noise sensitivity, identifying the best techniques that can be applied in experimental analysis. Signals are generated by numerical integration of the nonlinear pendulum mathematical model, selecting a single variable of the system as a time series. In order to simulate experimental data sets, a random noise is introduced in the time series. State space reconstruction is done employing the method of delay coordinates. The determination of delay parameters, time delay and embedding dimension, are made employing, respectively, the method of average mutual information and the false nearest neighbors. Both methods present good results and are not noise sensitive. Concerning the attractor dimension, one employs the correlation dimension discussed by Hegger *et al*. (1999), based on the Theiler's algorithm, and also the generalized dimension developed by Sarraille & Myers (1994). The values are compared with a reference value that is achieved employing the Lyapunov dimension estimated by numerical simulation. Results show that the dimension of the attractor is not an efficient tool to diagnose chaos. The value calculated with the algorithm due to Hegger *et al.* (1999) is sensitive to the embedding dimension and, even though the noise does not have a significantly influence, it is difficult to obtain conclusive results. The algorithm due to Sarraille & Myers (1994), on the other hand, estimates values which are closer than the reference and also is not significantly influenced by the noise. Nevertheless, its use needs large data points because the convergence occurs for high values of embedding dimension. Hence, it is unfeasible to apply this procedure on the analysis of experimental data. The authors agree that this contribution is useful to identify the best techniques that may be applied in experimental analysis, however, the investigation of other physical systems and different kinds of noise are necessary to assure these conclusions.

Acknowledgments

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