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Chaos in a Two-Degree of Freedom Duffing Oscillator

High dimensional dynamical systems has intricate behavior either on temporal or on spatial evolution properties. Nevertheless, most of the work on chaotic dynamics has been concentrated on temporal behavior of low-dimensional systems. This contribution is concerned with the chaotic response of a two-degree of freedom Duffing oscillator. Since the equations of motion are associated with a five-dimensional system, the analysis is performed by considering two Duffing oscillators, both with single-degree of freedom, coupled by a spring-dashpot system. With this assumption, it is possible to analyze the transmissibility of motion between the two oscillators.
***Keywords**: Chaos, duffing oscillator*

Introduction

Nonlinear dynamics of mechanical systems presents some characteristics not observed in linear systems. As an example one could mention chaotic motion where unpredictability and sensitivity to initial conditions are some important characteristics. The study of chaos considers proper mathematical and geometrical aspects. Therefore, new analytical, computational and experimental methods are developed to analyze the nonlinear response of dynamical systems. Since these aspects usually consider geometrical approach, they introduce difficulties to describe systems with many degrees of freedom (Alligood *et al.*, 1997; Moon, 1992; Hilborn, 1994; Mullin, 1993; Ott, 1993; Kapitaniak, 1991; Wiggins, 1990; Schuster, 1989; Thompson & Stewart, 1986; Guckenheimer & Holmes, 1983). High dimensional dynamical systems have intricate behavior either on temporal or on spatial evolution properties. In the past, most of the work on chaotic dynamics has been concentrated on temporal behavior of low-dimensional systems. Recently, spatiotemporal chaos has attracted much attention due to its theoretical and practical applications (Savi & Pacheco, 2002; Lai & Grebogi, 1999; Shibata, 1998; Barreto *et al.*, 1997; Thompson & Van der Heijden, 1997; Umberger *et al.*, 1989).

Many researches have been developed to study dynamical systems described by simple mathematical models. Despite the deceiving simplicity of these models, their nonlinear dynamic response may exhibit a number of interesting, complex behaviors. Mathematically, there are two kinds of dynamical models: *differential equations model*, which is continuous in time, and *map*, which describes the time evolution of a system by expressing its state as a function of its previous time. Therefore, map is a dynamical system moving through time in discrete updates. One of the most important uses of maps is to assist in the study of a differential equation model (Alligood *et al.*, 1997). Duffing and van der Pol oscillators, nonlinear pendulum and Lorenz system are some examples of classical dynamical systems described by differential equations model (Guckenheimer & Holmes, 1983). On the other hand, logistic and tent map are some of the one-dimensional maps while Henon and Ikeda maps are some of the classical two-dimensional maps (Alligood *et al.*, 1997; Ott, 1993).

The Duffing oscillator has been used to model the nonlinear dynamics of special types of mechanical and electrical systems. The differential equation that describes this oscillator has a cubic nonlinearity, and it has been named after the studies of G. Duffing in the 1930's. A number of physical systems can be described using Duffing equation. As some examples, one could mention an electrical circuit with a nonlinear inductor and the postbuckling vibrations of an elastic beam column under compressive loads.

This contribution discusses the nonlinear dynamics of a Duffing oscillator with two-degree of freedom. The prospect of chaotic behavior is of concerned and, since the equations of motion are associated with a five-dimensional system, the analysis is performed by considering two Duffing oscillators, both with single-degree of freedom, coupled by a spring-dashpot system. With this assumption, it is possible to analyze the transmissibility of motion between the two oscillators. Numerical results allow one to obtain conclusions that may be used to understand the behavior of other dynamical system. The analysis of the projection of the five-dimensional phase space in three-dimensional plots, shows some characteristics of the phase space projection.

Equations of Motion

Consider a two-degree of freedom oscillator, depicted in Fig.1, which consists of two masses, m_i (i = 1,2), supported by nonlinear springs with stiffness K_i (i = 1,2,3) and linear dampers with coefficient c_i (i = 1,2,3). The system is harmonically excited by two forces $F_i = \delta_i \sin(\Omega_i t)$ (i = 1,2).

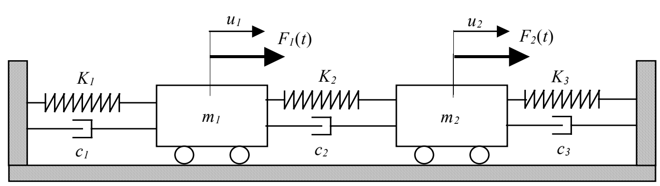

Figure 1 - Two-degree of freedom Duffing oscillator.

The nonlinear spring behavior is described by considering that the restoring force is not linearly proportional to the displacement. The behavior of each spring is described by the following function, where a cubic nonlinearity is considered,

$$K_i = K_i(u) = k_i u + a_i u^3 \qquad (1)$$

The variable u represents the displacement associated with the spring; k_i and a_i are constants. By establishing the equilibrium of the system and assuming $y_0 = u_1$, $y_1 = \dot{u}_1$, $y_2 = u_2$ and $y_3 = \dot{u}_2$, the following dynamical system is written

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$$
\begin{aligned}
\dot{y}_0 &= y_1 \\
\dot{y}_1 &= (1/m_1)[F_1(t) - (c_1 + c_2)y_1 + c_2 y_3 - (k_1 + k_2)y_0 + k_2 y_2 - a_1 y_0^3 + a_2(y_2 - y_0)^3] \\
\dot{y}_2 &= y_3 \\
\dot{y}_3 &= (1/m_2)[F_2(t) + c_2 y_1 - (c_2 + c_3)y_3 + k_2 y_0 - (k_2 + k_3)y_2 - a_2(y_2 - y_0)^3 - a_3 y_2^3]
\end{aligned} \tag{2}
$$

The characterization of chaotic motion must be considered by some criterion, which establishes a quantitative definition of chaos. Lyapunov exponents are an acceptable criterion used in this article and its estimation is done employing the algorithm proposed by Wolf *et al.* (1985).

In the following sections, numerical simulations of forced response of the Duffing oscillator are discussed. In all simulations, one has taken $m_1 = m_2 = 1$, $k_1 = k_3 = -0.02$, $a_1 = a_3 = 1$, $c_1 = c_3 = 0.05$. A harmonic excitation $F_i(t) = \delta_i \sin(\Omega_i t)$ (i = 1,2) is assumed. Numerical simulations are performed employing a fourth-order Runge-Kutta scheme with time steps chosen to be less than $\Delta t = 2\pi/600\Omega$.

Linear Connection

In this Section, two Duffing oscillators, both with single-degree of freedom, coupled by a linear spring-viscous damper system are considered. Therefore, one considers $a_2 = 0$, and the parameters c_2 and k_2 are used to analyze the transmissibility of motion between the two oscillators. By considering $c_2 = k_2 = 0$, it is clear that there are two uncoupled oscillators. The same frequency parameter is taken for both oscillators, $\Omega_1 = \Omega_2 = 1$, while two different forcing amplitudes are assumed: $\delta_1 = 7.5$ and $\delta_2 = 4$. The parameter $\delta_1 = 7.5$ causes chaotic motion on the first oscillator while $\delta_2 = 4$ results in a periodic motion (Fig.2).

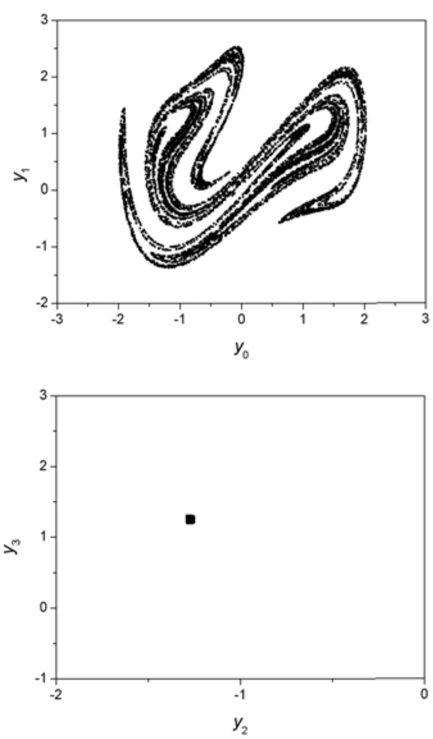

Figure 2. Poincaré section for two oscillators ($k_2 = c_2 = 0$).

In order to start the analysis of transmissibility of motion, a linear spring connection is considered in order to couple both oscillators. Therefore $c_2 = 0$, and the parameter k_2 may vary. First, bifurcation diagrams are considered to represent the stroboscopically sampled displacement values, y_0 and y_2, under the slow quasi-static increase of parameter k_2 (Fig.3). The chaotic motion of mass m_1 is transmitted to mass m_2 when k_2 decreases, however, the Poincaré section associated with mass m_2 has a different pattern from the usual form of the strange attractor presented by the mass m_1 (Fig.4).

Fig. 5 shows the 3D plot (y_0-y_1-y_2) of the five-dimensional phase space. The attractor of mass m_1 (Fig. 4) can be saw in the projection on the y_0-y_1 plane. Observing the projections on the y_1-y_2 and y_0-y_2 planes, it is possible to see that the attractor does not present a cantor set like structure on the direction of the variable y_2. The same behavior is observed for 3D plots considering the variables y_0-y_1-y_3.

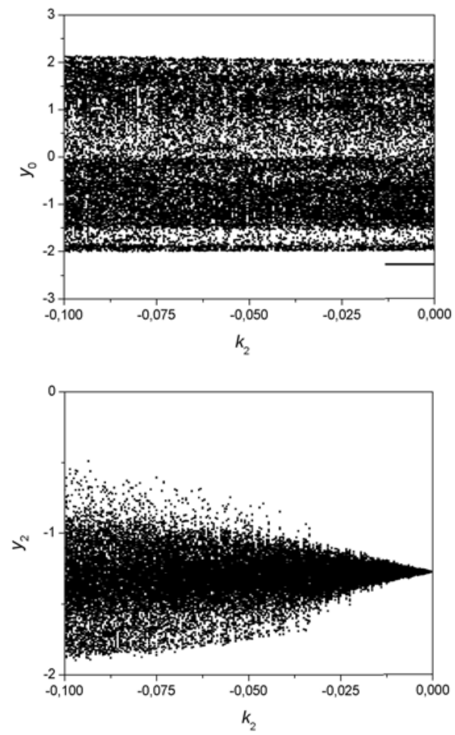

Figure 3. Bifurcation diagrams for y_0 vs k_2 and y_2 vs k_2 with $c_2 = 0$.

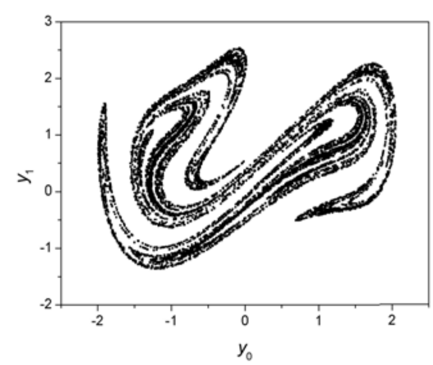

Figure 4. Poincaré section with $c_2 = 0$ and $k_2 = -0.02$.

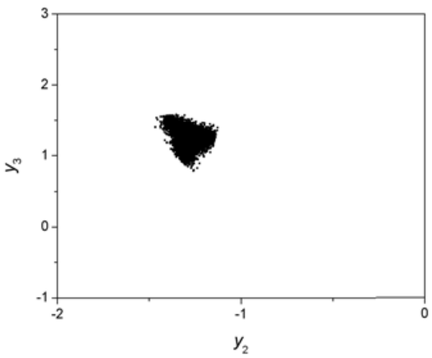

Figure 4. (Continued).

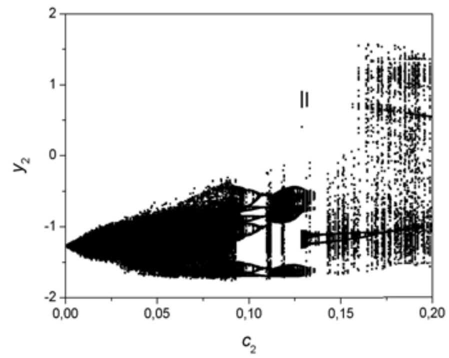

Figure 6. (Continued).

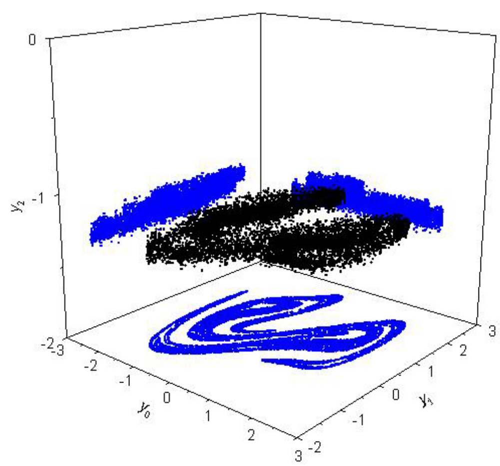

Figure 5. Poincaré section in the space y_0-y_1-y_2 for c_2=0 and k_2 = –0.02.

Now, a linear viscous damper connection is focused. Therefore, k_2 = 0, and the parameter c_2 is used to analyze the transmissibility of motion between the two oscillators. Bifurcation diagrams relating the sampled displacement values, y_0 and y_2, under the slow quasi-static increase of parameter c_2 are considered (Fig.5). The energy dissipation on the connection establishes a different kind of transmissibility. For high values of the parameter c_2, the motion of the two masses tends to be similar.

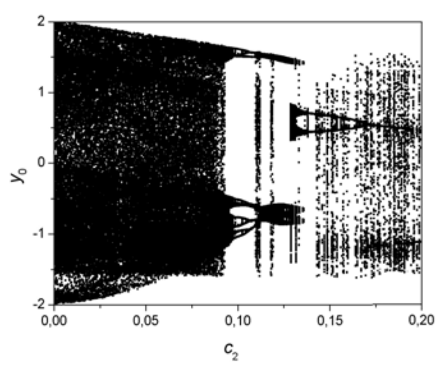

Figure 6. Bifurcation diagrams for y_0 vs c_2 and y_2 vs c_2 with k_2 = 0.

Fig.7-8 show the Poincaré sections for k_2 = 0 and some different values of the dissipation parameter. Fig.7 considers c_2 = 0.05, and chaotic motion is observed on both masses. Again, the strange attractor of mass m_1 has a usual form, while mass m_2 presents a different pattern. Fig.8 considers c_2 = 0.1, and there is a periodic motion.

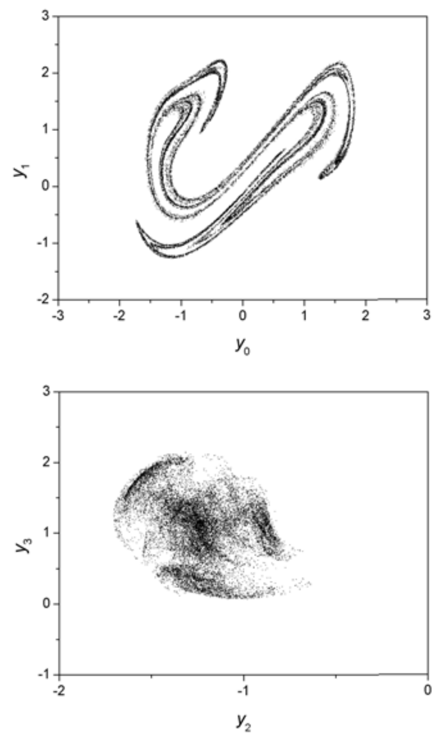

Figure 7. Poincaré section with k_2 = 0 and c_2 = 0.05.

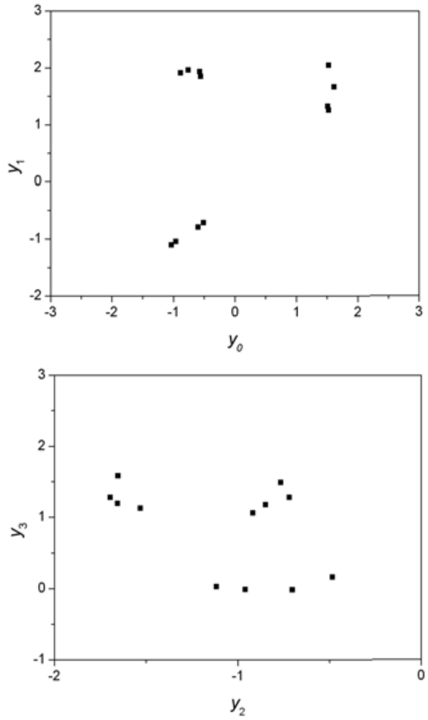

Figure 8. Poincaré section with $k_2 = 0$ and $c_2 = 0.1$.

Fig. 9 shows the 3D plot (y_0-y_1-y_2) of the five-dimensional phase space discussed in Fig.7. The attractor of mass m_1 (Fig.7) can be saw in the projection on the y_0-y_1 plane. The system presents the same behavior observed in Fig.5, even though the patterns of the projections on the other planes are not similar to the one presented in Fig.5.

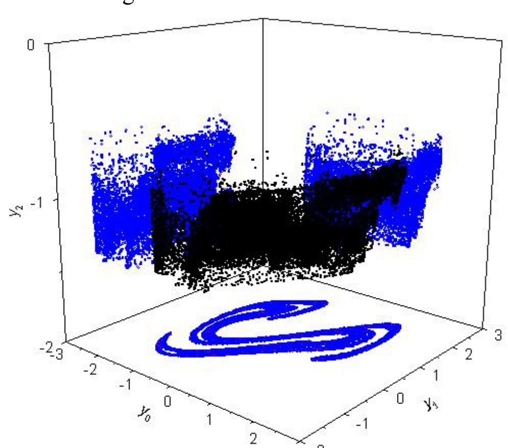

Figure 9. Poincaré section in the space y_0-y_1-y_2 for c_2=0.05 and $k_2 = 0$.

A linear spring-viscous damper connection is now considered. The spring constant is $k_2 = -0.02$, and the parameter c_2 is used to analyze the transmissibility of motion between the two oscillators. Fig.10 shows the bifurcation diagrams for this situation.

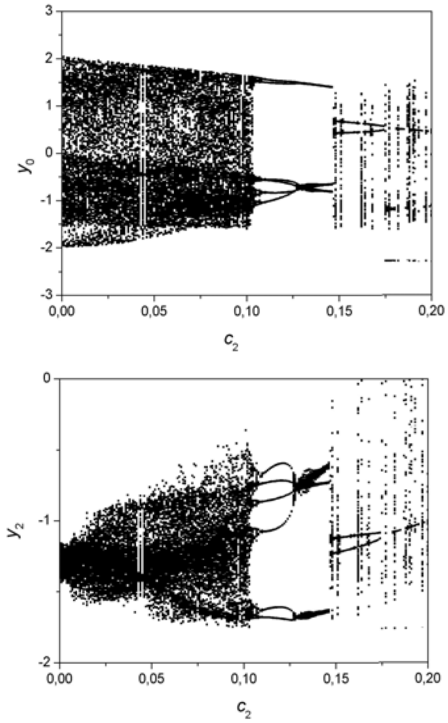

Figure 10. Bifurcation diagrams for y_0 vs c_2 and y_2 vs c_2 with $k_2 = -0.02$.

Fig.11-12 show the Poincaré sections for $k_2 = -0.02$ and some different values of the dissipation parameter. Fig.11 considers $c_2 = 0.05$, while Fig.12 considers $c_2 = 0.1$. For both cases, chaotic motion is observed in both masses, and again, the Poincaré section associated with mass m_2 present a different pattern of strange attractor.

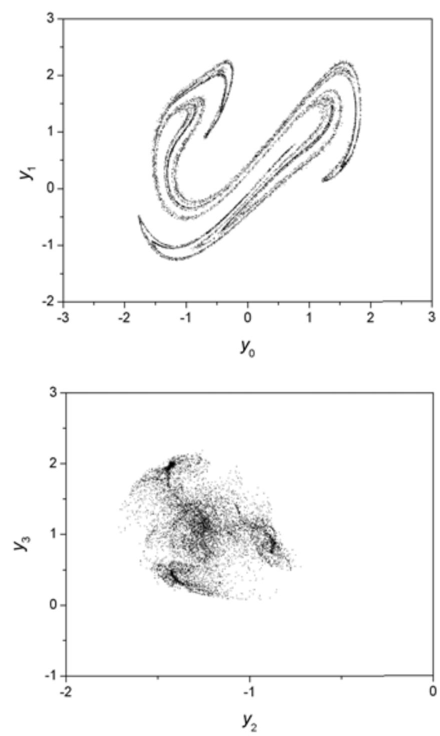

Figure 11. Poincaré section with $k_2 = -0.02$ and $c_2 = 0.05$.

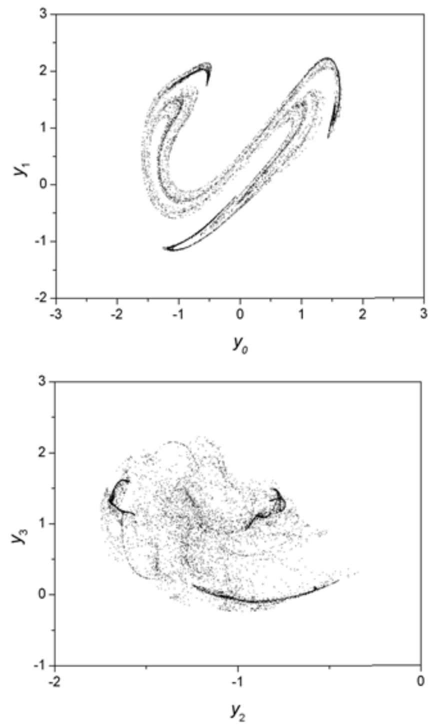

Figure 12. Poincaré section with k_2 = –0.02 and c_2 = 0.1.

Fig.13-14 shows the 3D plot (y_0-y_1-y_2) of the five-dimensional phase space discussed in Fig.11-12. Notice that the attractor of mass m_1 can be observed in the projection on the y_0-y_1 plane. It should be pointed out that in Fig.14, the patterns of the projections on the other planes are not similar to the others.

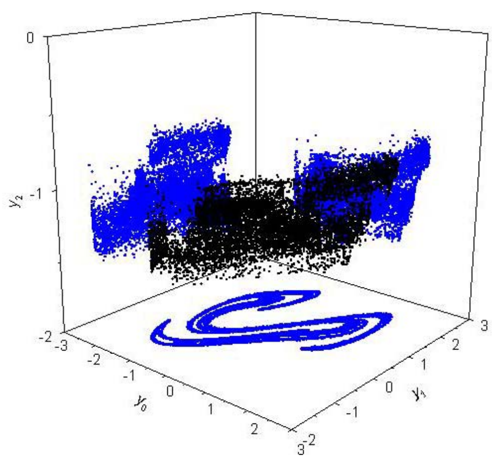

Figure 13. Poincaré section in the space y_0-y_1-y_2 for c_2=0.05 and k_2 = –0.02.

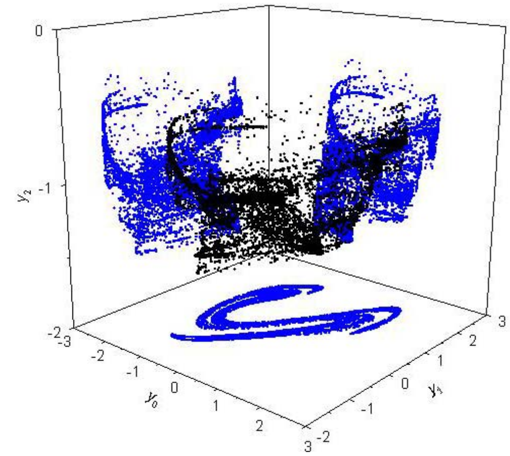

Figure 14. Poincaré section in the space y_0-y_1-y_2 for c_2=0.1 and k_2 = –0.02.

Nonlinear Connection

In this section, a nonlinear connection between the two Duffing oscillators, both with single-degree of freedom, is investigated. Therefore, one considers a_2 = 1, and the parameters c_2 e k_2 are used to analyze the transmissibility of motion between the two oscillators. The same conditions presented on the preceding section are taken. In order to start the analysis, one considers bifurcation diagrams relating the sampled displacement values, y_0 and y_2, under the slow quasi-static increase of parameter k_2. It is also assumed that there is no dissipation on the connection, c_2 = 0 (Fig.15).

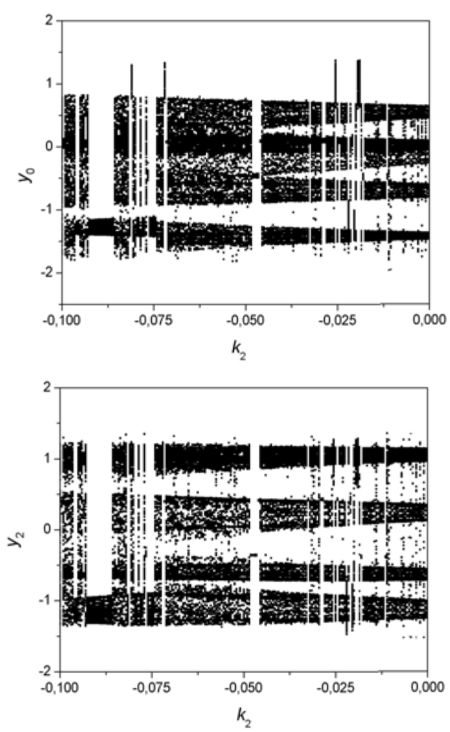

Figure 15. Bifurcation diagrams for y_0 vs k_2 and y_2 vs k_2 with a_2 = 1 and c_2 = 0.

Fig.16-17 show the Poincaré sections for two different sets of parameters. For these examples, a different pattern of chaotic motion occurs for the two masses.

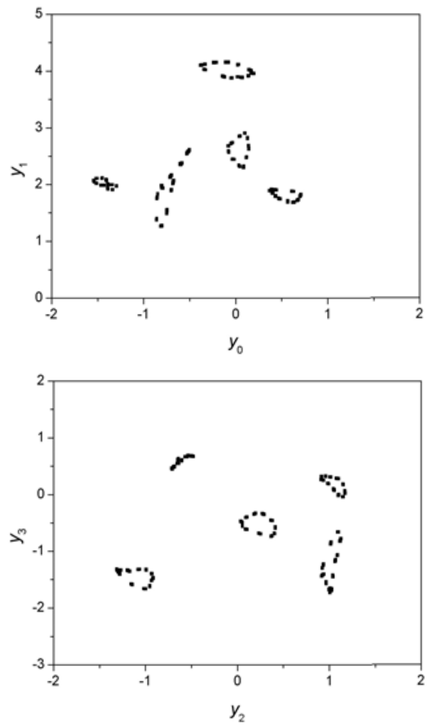

Figure 16. Poincaré sections with a_2 = 1, c_2 = 0 and k_2 = –0.0375.

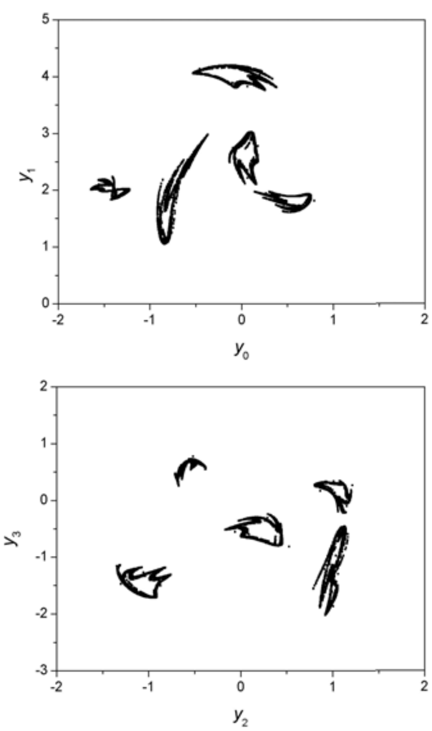

Figure 17. Poincaré sections with a_2 = 1, c_2 = 0 and k_2 = –0.0625.

Observing the 3D plot (y_0-y_1-y_2) of the five-dimensional phase space discussed in Fig.17, it is possible to see the unusual form of this attractor (Fig.18).

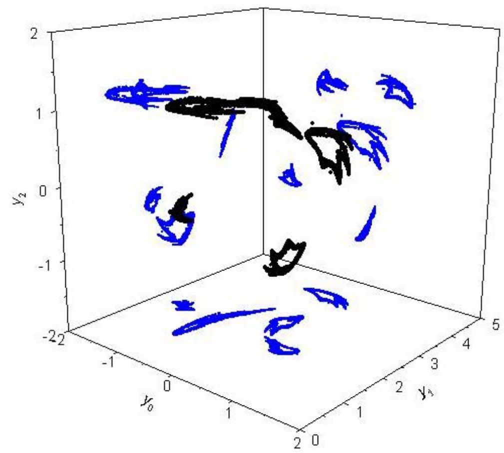

Figure 18. Poincaré section in the space y_0-y_1-y_2 for a_2 = 1, c_2 = 0 and k_2 = –0.0625.

A nonlinear spring-viscous damper connection is in order. The spring constant is k_2 = –0.02, and the parameter c_2 is used to analyze the transmissibility of motion between the two oscillators. Fig.19 shows the bifurcation diagrams for this situation.

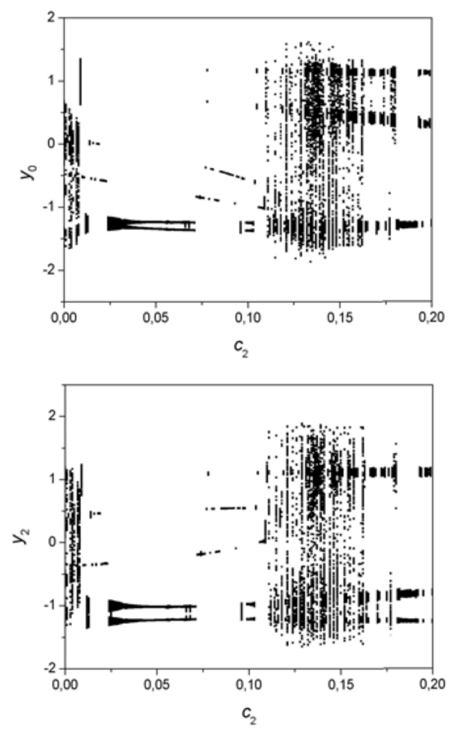

Figure 19. Bifurcation diagrams for y_0 vs k_2 and y_2 vs k_2 with a_2 = 1 and k_2 = –0.02.

By considering a_2 = 1, k_2 = –0.02 and c_2 = 0.05, a period-2 response occurs. Fig.20 shows a typical Poincaré section for this situation.

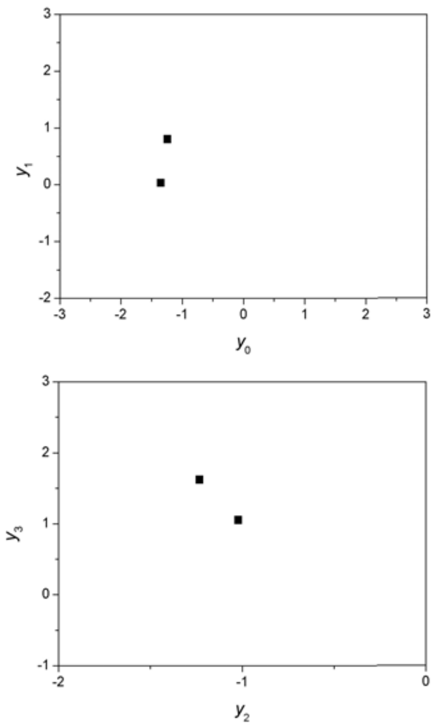

Figure 20. Poincaré section with a_2 = 1, k_2 = –0.02 and c_2 = 0.05.

Conclusions

This contribution discusses the chaotic response of a two-degree of freedom Duffing oscillator. Numerical simulations are obtained using the fourth order Runge-Kutta method. Since equations of motion are associated with a five-dimensional system, the analysis is performed by considering two Duffing oscillators, both with single-degree of freedom, coupled by a spring-dashpot system. With this assumption, it is possible to analyze the transmissibility of motion between the two oscillators. Results show that chaotic motion of one mass is transmitted with different patterns to the other mass and reveals that a very complex behavior can be expected for other dynamical system either with multiple degrees of freedom or continuous.

Acknowledgements

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