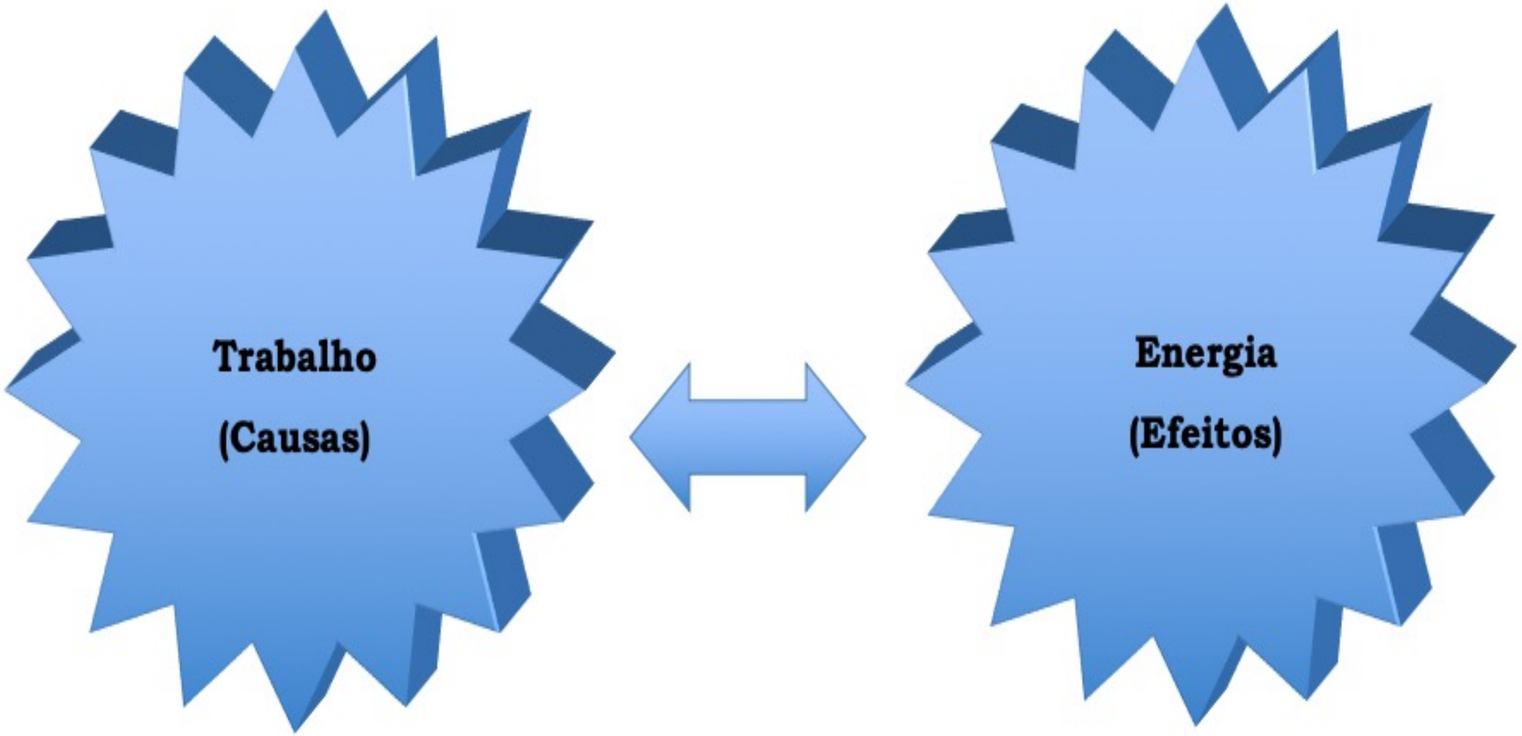
DINÂMICA: Abordagem Energética

Marcelo A. Savi

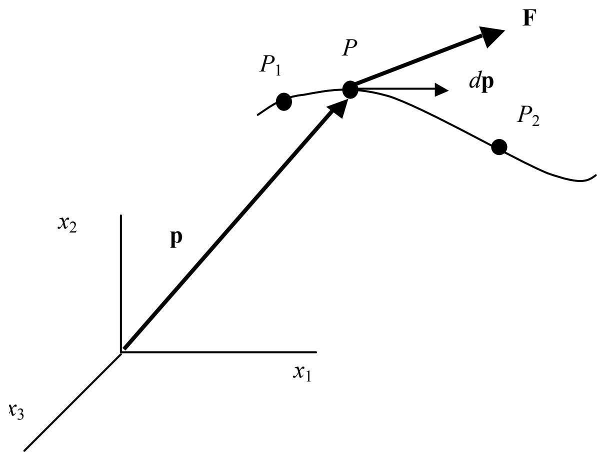
Universidade Federal do Rio de Janeiro

Abordagem Energética

- A abordagem energética ou mecânica analítica estabelece a descrição quadro-a-quadro da realidade a partir de conceitos energéticos.



Trabalho



$$\mathcal{T} = \int_{P_1}^{P_2} \mathbf{F}.\, d\mathbf{p}^P$$

$$\mathcal{T} = \int_{t_1}^{t_2} \mathbf{F}.\frac{d\mathbf{p}^P}{dt} dt = \int_{t_1}^{t_2} \mathbf{F}.\, \mathbf{v}^P dt$$

Trabalho de um sistema de forças:

$$\sum_{i=1}^{N} \mathcal{T}^{\mathbf{F}_i} = \sum_{i=1}^{N} \int_{P_1}^{P_2} \mathbf{F}_i.\, d\mathbf{p}^P = \int_{P_1}^{P_2} \sum_{i=1}^{N} \mathbf{F}_i.\, d\mathbf{p}^P = \int_{P_1}^{P_2} \mathbf{R}.\, d\mathbf{p}^P = \mathcal{T}^{\mathbf{R}}$$

Forças conservativas – associadas e uma energia potencial:

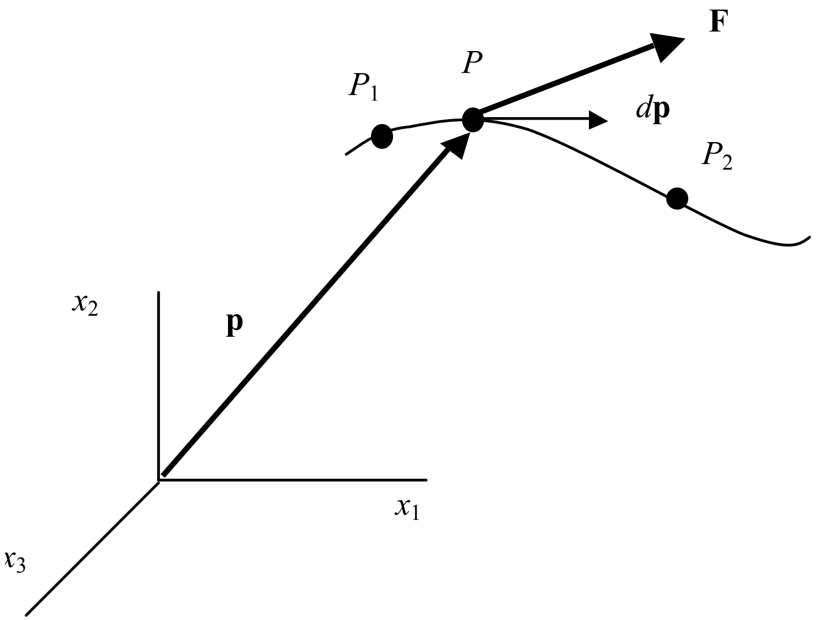
$$\mathbf{F}_C = -\nabla E_P$$

Trabalho de forças conservativas e não-conservativas:

$$\mathcal{T} = \mathcal{T}_{\mathrm{C}} + \mathcal{T}_{\mathrm{N}}$$

$$\mathcal{T}_{\mathrm{C}} = \int_{P_1}^{P_2} \mathbf{F}_C.\, d\mathbf{p}^P = \int_{P_1}^{P_2} -\nabla E_P.\, d\mathbf{p}^P = E_P^{P_1} - E_P^{P_2}$$

Energia Cinética



Energia é a capacidade de realizar trabalho:

$$d\mathcal{T} = \mathbf{F}\cdot d\mathbf{p}^P = m\dot{\mathbf{v}}^P\cdot d\mathbf{p}^P = m\dot{\mathbf{v}}^P\cdot \mathbf{v}^P dt$$
$$= \frac{d}{dt}\left(\frac{1}{2}m\mathbf{v}^P\cdot\mathbf{v}^P\right)dt = dE_C$$

Mas: $$\frac{d}{dt}\left(\frac{1}{2}m\mathbf{v}^P\cdot\mathbf{v}^P\right) = m\dot{\mathbf{v}}^P\cdot\mathbf{v}^P$$

$$E_C = \int_{t_1}^{t_2} m\dot{\mathbf{v}}^P\cdot\mathbf{v}^P dt = \int_{t_1}^{t_2}\dot{\mathbf{G}}^P\cdot\mathbf{v}^P dt$$

$$E_C = \frac{1}{2}m\mathbf{v}^P.\mathbf{v}^P = \frac{1}{2}mv_i^P v_i^P$$

Conservação de Energia

Segunda lei de Newton: $\dot{\mathbf{G}}^P = m\ddot{\mathbf{p}}^P = m\mathbf{a}^P = \mathbf{F}$

$$m\mathbf{a}^P . d\mathbf{p}^P = \mathbf{F}. d\mathbf{p}^P$$

Note que: $\mathbf{a}^P . d\mathbf{p}^P = m\frac{d\mathbf{v}^P}{dt}. d\mathbf{p}^P = m\frac{d\mathbf{v}^P}{dt}.\frac{d\mathbf{p}^P}{dt}dt = m\frac{d\mathbf{v}^P}{dt}.\mathbf{v}^P dt = m\mathbf{v}^P . d\mathbf{v}^P = \mathbf{G}^P . d\mathbf{v}^P$

Portanto:

$$\boxed{\int_{P_1}^{P_2} d\mathbf{G}^P . d\mathbf{v}^P = \int_{P_1}^{P_2} \mathbf{F}. d\mathbf{p}^P}$$

O trabalho das forças externas é igual a variação da energia cinética.

$$\boxed{E_C^{P_2} - E_C^{P_1} = \mathcal{T}}$$

Conservação de Energia

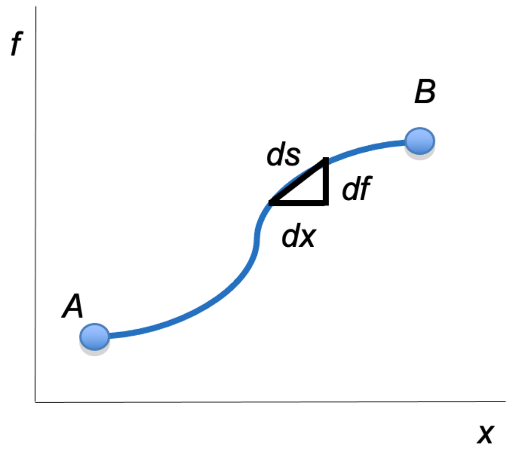
Usando a ideia de forças conservativas: $\left(E_C^{P_2} - E_C^{P_1}\right) + \left(E_P^{P_2} - E_P^{P_1}\right) = \mathcal{T}_N$

Energia mecânica: $E = E_C + E_P$

Tem-se a conservação de energia mecância: $E^{P_2} - E^{P_1} = \mathcal{T}_N$

Lagrangeano: $\mathcal{L} = E_C - E_P$

Cálculo Variacional:
Distância Entre Dois Pontos



$$ds^2 = dx^2 + df^2 = \left[1 + \left(\frac{df}{dx}\right)^2\right] dx^2$$

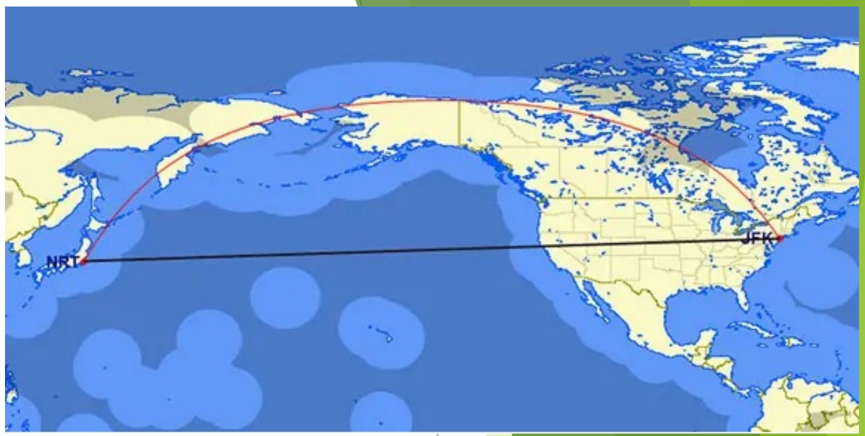
$$ds = \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx$$

Funcional:

$$\mathcal{H} = \int_A^B ds = \int_A^B \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx$$

$$\mathcal{H} = \int_A^B \hbar\left(x, f(x), \frac{df}{dx}(x)\right) dx$$

A determinação da menor distância entre os dois pontos passa pela minimização do funcional.



Funcional de Energia

Funcional de energia depende do tempo, da posição e da velocidade que dependem de coordenadas generalizadas $p = p(q)$:

$$\mathcal{H} = \int_A^B \hbar(t, q, \dot{q})dt$$

Considerando a ideia de trabalho e energia:

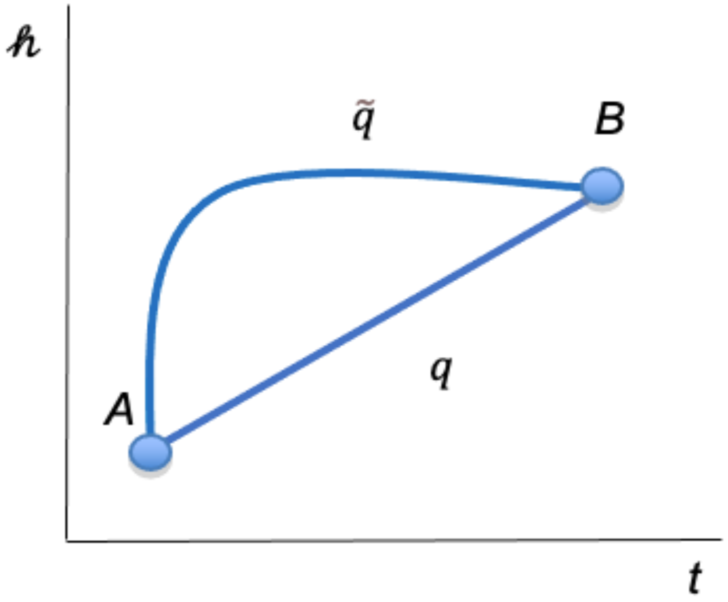
$$\int_{t_1}^{t_2} (\mathbf{G}^P \cdot \dot{\mathbf{v}}^P + \nabla E_P \cdot \dot{\mathbf{p}}^P)dt = 0$$

$$\mathcal{H} = \int_{t_1}^{t_2} \mathcal{L}(t, q, \dot{q})\, dt$$

Operador Variacional

Funcional:

$$\mathcal{H} = \int_A^B \hbar(t, q, \dot{q})dt$$



Função perturbada:

$$\tilde{q} = q + \delta q$$

$$\delta q = \tilde{q} - q = \varepsilon \eta$$

Minimização de Funcionais

Funcional perturbado:

$$\tilde{\mathcal{H}} = \int_A^B \tilde{\hbar}(t, \tilde{q}, \dot{\tilde{q}})dt = \int_A^B \tilde{\hbar}(t, q + \delta q, \dot{q} + \delta \dot{q})dt$$

Série de Taylor:

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}\Big|_{\varepsilon=0} + \frac{\partial \tilde{\mathcal{H}}}{\partial \varepsilon}\Big|_{\varepsilon=0} \varepsilon + \frac{1}{2!}\frac{\partial^2 \tilde{\mathcal{H}}}{\partial \varepsilon^2}\Big|_{\varepsilon=0} \varepsilon^2 + \cdots$$

$$\tilde{\mathcal{H}} = \mathcal{H} + \delta\mathcal{H} + \delta^2\mathcal{H} + \cdots$$

Condição de estacionalidade:

$$\delta\mathcal{H} = \tilde{\mathcal{H}} - \mathcal{H} = 0$$

Minimização de Funcionais

Para avaliar $\delta\mathcal{H} = \tilde{\mathcal{H}} - \mathcal{H} = 0$:

$$\delta\mathcal{H} = \left.\frac{\partial\tilde{\mathcal{H}}}{\partial\varepsilon}\right|_{\varepsilon=0} \varepsilon = \varepsilon \frac{\partial}{\partial\varepsilon}\left[\int_A^B \tilde{\hbar}(t,\tilde{q},\dot{\tilde{q}})dt\right]_{\varepsilon=0} = \int_A^B \left[\frac{\partial\tilde{\hbar}}{\partial\tilde{q}}\frac{\partial\tilde{q}}{\partial\varepsilon} + \frac{\partial\tilde{\hbar}}{\partial\dot{\tilde{q}}}\frac{\partial\dot{\tilde{q}}}{\partial\varepsilon}\right]\varepsilon dt$$

onde

$$\frac{\partial\tilde{q}}{\partial\varepsilon} = \frac{\partial}{\partial\varepsilon}(q + \varepsilon\eta) = \eta \qquad \left.\frac{\partial\tilde{\hbar}}{\partial\tilde{q}}\right|_{\varepsilon=0} = \frac{\partial\hbar}{\partial q}$$

$$\frac{\partial\dot{\tilde{q}}}{\partial\varepsilon} = \frac{\partial}{\partial\varepsilon}(\dot{q} + \varepsilon\dot{\eta}) = \dot{\eta} \qquad \left.\frac{\partial\tilde{\hbar}}{\partial\dot{\tilde{q}}}\right|_{\varepsilon=0} = \frac{\partial\hbar}{\partial\dot{q}}$$

Minimização de Funcionais

$$\delta\mathcal{H} = \int_A^B \left[\frac{\partial \hbar}{\partial q}\eta + \frac{\partial \hbar}{\partial \dot{q}}\dot{\eta}\right]\varepsilon dt = \int_A^B \left[\frac{\partial \hbar}{\partial q}\delta q + \frac{\partial \hbar}{\partial \dot{q}}\delta \dot{q}\right] dt$$

Integração por partes:

$$\int_A^B \frac{\partial \hbar}{\partial \dot{q}}\delta\dot{q}dt = \frac{\partial \hbar}{\partial \dot{q}}\delta q\Big|_A^B - \int_A^B \frac{d}{dt}\left(\frac{\partial \hbar}{\partial \dot{q}}\right)\delta q dt$$

$$\delta\mathcal{H} = \int_A^B \left[\frac{\partial \hbar}{\partial q} - \frac{d}{dt}\left(\frac{\partial \hbar}{\partial \dot{q}}\right)\right]\delta q dt$$

Condições de estacionalidade: $\delta\mathcal{H} = 0$

$$\frac{d}{dt}\left(\frac{\partial \hbar}{\partial \dot{q}}\right) - \frac{\partial \hbar}{\partial q} = 0$$

Equações de Lagrange

Princípio de Hamilton

$$\mathcal{H} = \int_{t_1}^{t_2} \left(\mathbf{G} \cdot \frac{d\dot{\mathbf{q}}}{dt} - \mathbf{F} \cdot \dot{\mathbf{q}} \right) dt$$

Usando o Lagrangeano:

$$\mathcal{H} = \int_{t_1}^{t_2} (\mathcal{L}(t, q, \dot{q}) - \mathbf{Q}^N \cdot \dot{\mathbf{q}}) dt$$

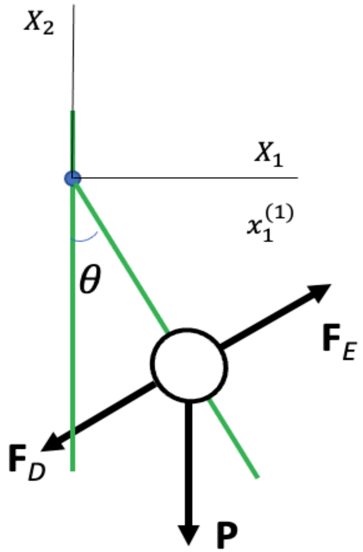
Princípio de Hamilton: minimização do functional

$$\delta\mathcal{H} = 0$$

O que resulta nas Equações de Lagrange:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = Q^N$$

Pêndulo



$$E_c = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\dot{\theta}^2$$

$$E_P = mgh = mgL(1 - c\theta)$$

$$\mathcal{L} = \frac{1}{2}mL^2\dot{\theta}^2 - mgL(1 - c\theta)$$

Forças generalizadas não conservativas : $Q^N = F(t) - \gamma L\dot{\theta}$

Equações de Lagrange: $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\right) - \frac{\partial \mathcal{L}}{\partial u} = Q^N$ onde $u = L\theta$

$$\frac{1}{L}\frac{\partial \mathcal{L}}{\partial \theta} = -mg\, s\theta$$

$$\frac{d}{dt}\left(\frac{1}{L}\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(mL\,\dot{\theta}\right) = mL\,\ddot{\theta}$$

$$mL\,\ddot{\theta} + \gamma L\dot{\theta} + mg\, s\theta = F(t)$$