DINÂMICA: Dinâmica da Partícula

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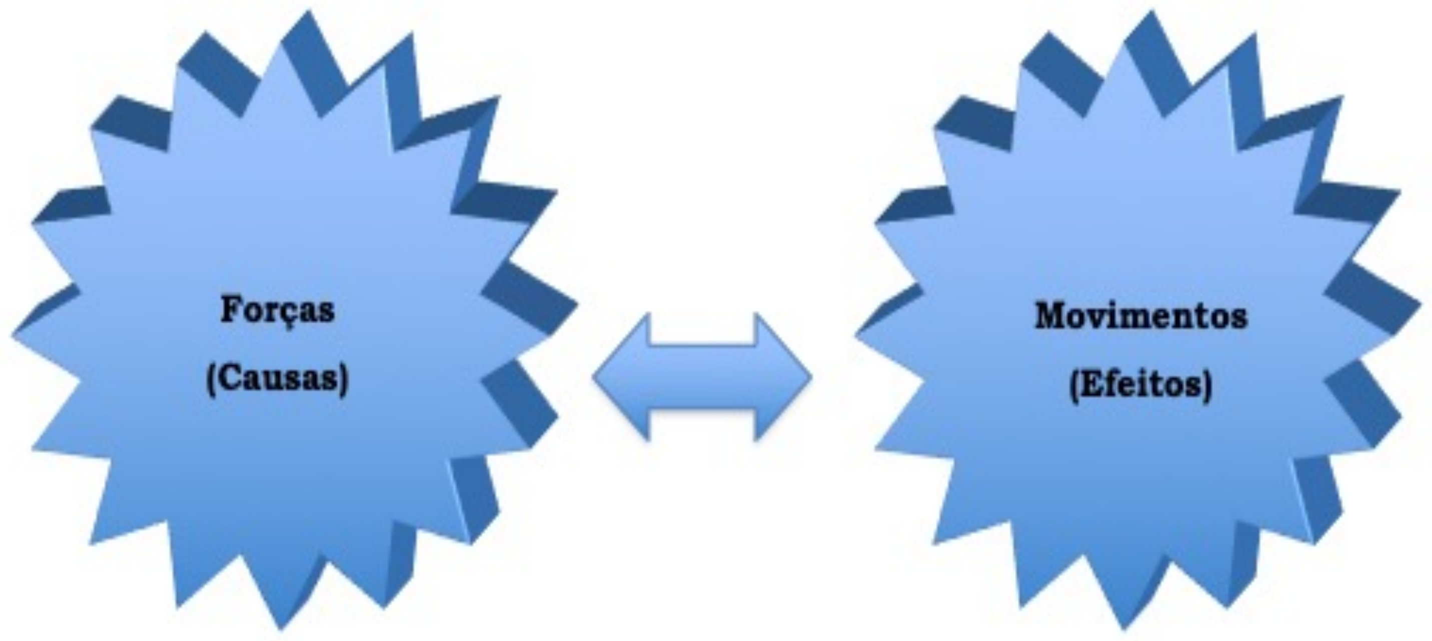
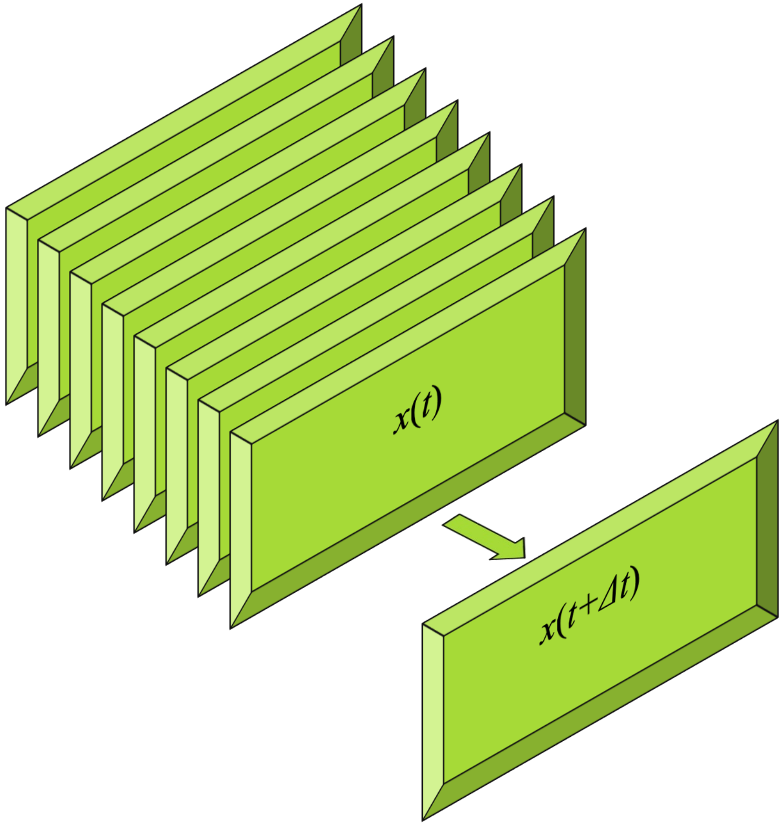
Dinâmica

- A dinâmica ou cinética define as equações de movimento que estabelece a descrição quadro-a-quadro da realidade.

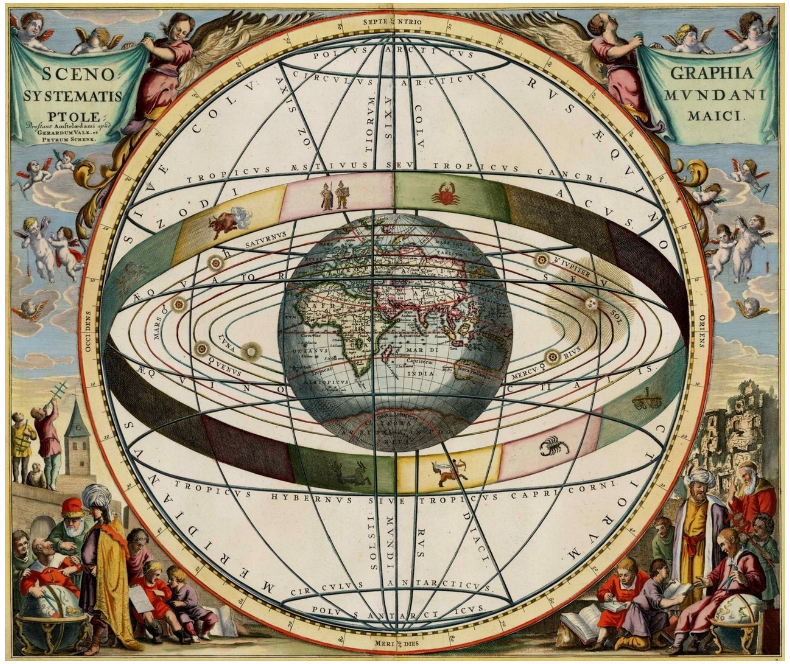


Equações Diferenciais:
Descrição quadro-a-quadro da realidade

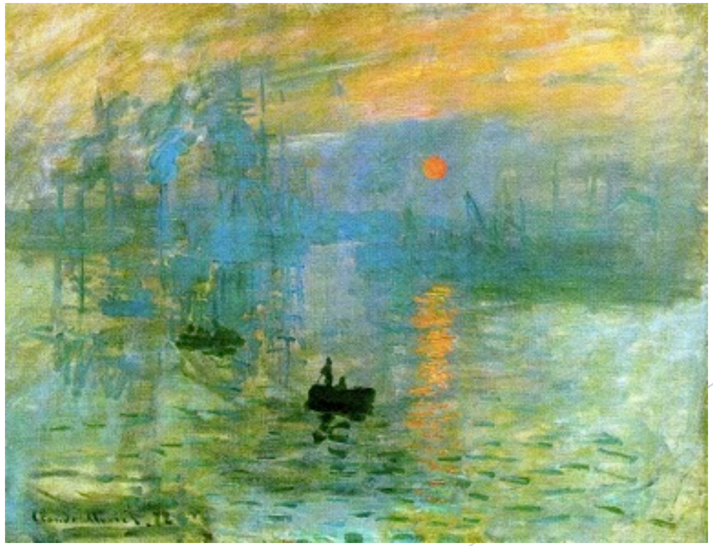


$$\dot{x} = f(x)\,,\ x \in R^n$$

Revolução Científica

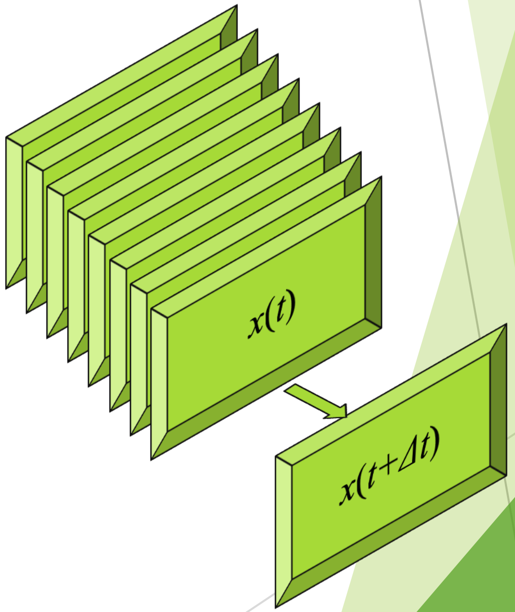


Modelando a Realidade



Modelando a Realidade:
Descrição quadro-a-quadro da realidade

$$\dot{x} = f(x)\,,\ \ x \in R^n$$

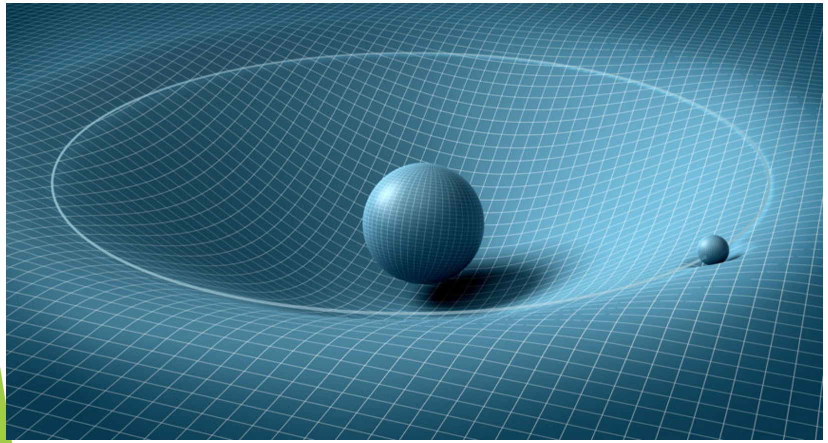


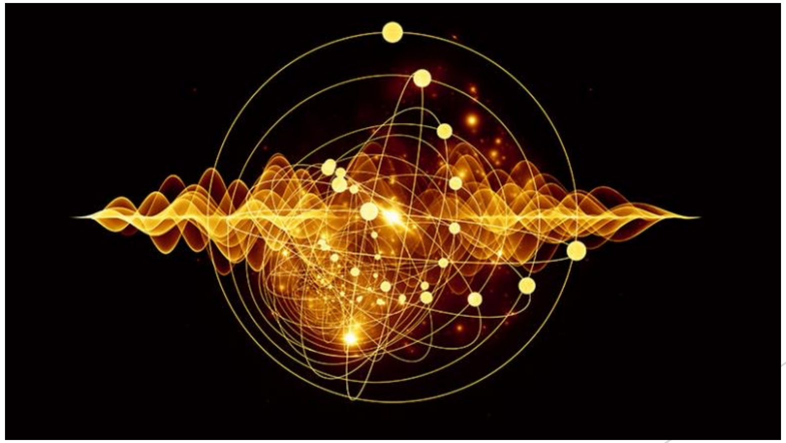
Evolução

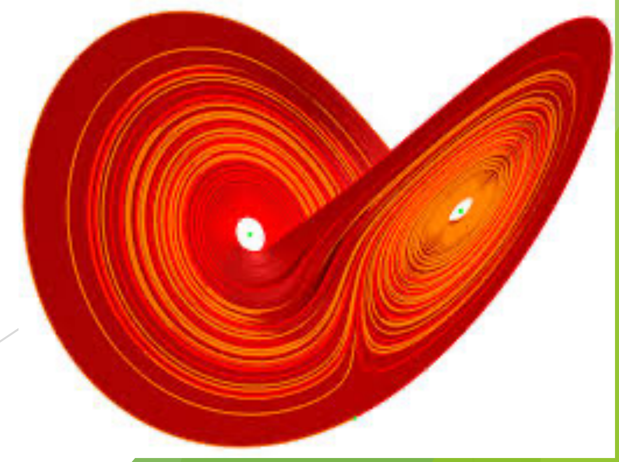
Sucesso do reducionismo e da descrição linear.

PROBLEMAS

- Teoria da Relatividade
- Mecânica Quântica
- Teoria do Caos







Leis de Newton

Primeira Lei:

Inércia - Se nenhuma força agir sobre a partícula, ela preserva a sua quantidade de movimento.

Segunda Lei:

Estabelece a relação entre força e movimento afirmando que a ação de uma força define a variação da quantidade de movimento.

$$\dot{\mathbf{G}}^P = \mathbf{F} \qquad m\ddot{\mathbf{p}}^P = m\mathbf{a}^P = \mathbf{F}$$

Terceira Lei:

Estabelece a relação entre dois corpos afirmando que, a toda ação em um corpo, corresponde uma reação de mesma intensidade e sentido contrário.

Equações de Movimento

$$\boxed{\dot{\mathbf{G}}^P = m\ddot{\mathbf{p}}^P = m\mathbf{a}^P = \mathbf{F}}$$

$$\mathbf{p}^P \times \dot{\mathbf{G}}^P = \mathbf{p}^P \times \mathbf{F}$$

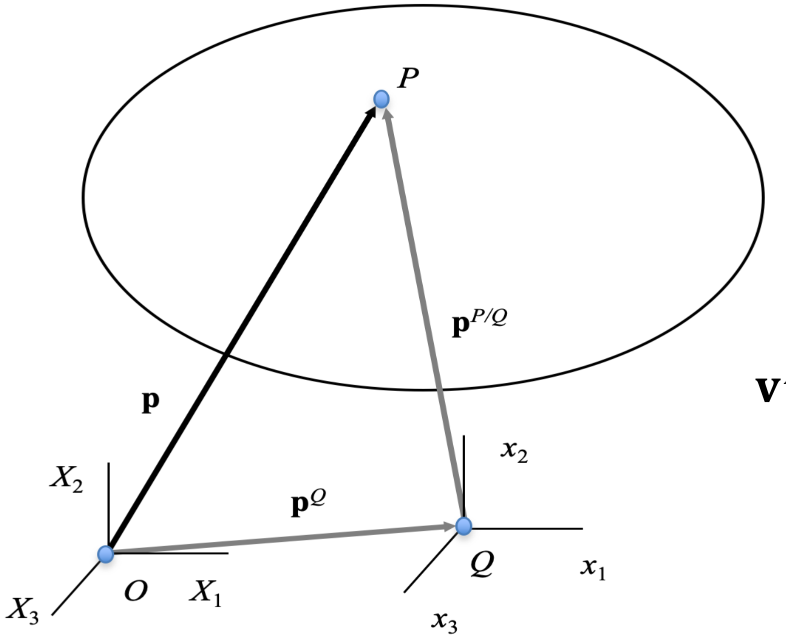
Mas $\mathbf{H}^{P/O} = \mathbf{p}^P \times \mathbf{G}^P$ $\quad \mathbf{M}^{\mathbf{F}/O} = \mathbf{p}^P \times \mathbf{F}$

$$\dot{\mathbf{H}}^{P/O} = \dot{\mathbf{p}}^P \times \mathbf{G}^P + \mathbf{p}^P \times \dot{\mathbf{G}}^P = \mathbf{p}^P \times \dot{\mathbf{G}}^P$$

$$\boxed{\dot{\mathbf{H}}^{P/O} = \mathbf{M}^{\mathbf{F}/O}}$$

0

$\dot{\mathbf{p}}^P = \mathbf{v}^P // \mathbf{G}^P$



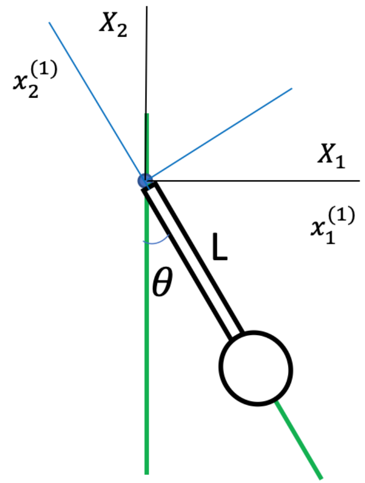
$$\mathbf{p}^{P/Q} \times \dot{\mathbf{G}}^P = \mathbf{p}^{P/Q} \times \mathbf{F}$$

$$\dot{\mathbf{H}}^{P/Q} = \mathbf{p}^{P/Q} \times \dot{\mathbf{G}}^P + \dot{\mathbf{p}}^{P/Q} \times \mathbf{G}^P = \mathbf{p}^{P/Q} \times \dot{\mathbf{G}}^P + \mathbf{v}^{P/Q} \times \mathbf{G}^P$$

$\mathbf{v}^{P/Q} = \mathbf{v}^P - \mathbf{v}^Q$ $\quad$ Mas $\mathbf{v}^P // \mathbf{G}^P$ $\quad$ $\dot{\mathbf{H}}^{P/Q} = \mathbf{p}^{P/Q} \times \dot{\mathbf{G}}^P - \mathbf{v}^Q \times \mathbf{G}^P$

$$\boxed{\dot{\mathbf{H}}^{P/Q} + \mathbf{v}^Q \times \mathbf{G}^P = \mathbf{M}^{\mathbf{F}/Q}}$$

Pêndulo



- Inercial, $I\ (X_i)$
- Móvel, solidário à haste, $S_1\ (x_i^{(1)})$

$$I \underset{\mathbf{T}_\theta^T}{\overset{\mathbf{T}_\theta}{\rightleftarrows}} S_1$$

$$\mathbf{T}_\theta = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{S_1}_{S_1}\mathbf{p}^P = \begin{Bmatrix} 0 \\ -L \\ 0 \end{Bmatrix}$$

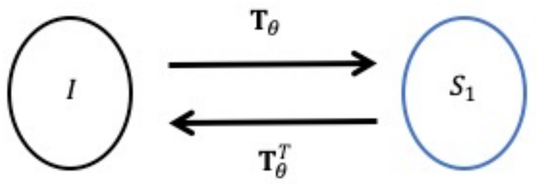
$${}^{I}_{S_1}\mathbf{a}^P = \cancelto{0}{{}^{I}_{S_1}\mathbf{a}^O} + \cancelto{0}{{}_{S_1}\mathbf{a}^{P/S_1}} + {}^{I}_{S_1}\boldsymbol{\alpha}^{S_1} \times {}_{S_1}\mathbf{p}^{P/S_1} + {}^{I}_{S_1}\boldsymbol{\omega}^{S_1} \times \left({}^{I}_{S_1}\boldsymbol{\omega}^{S_1} \times {}_{S_1}\mathbf{p}^{P/S_1} \right) + 2\,{}^{I}_{S_1}\boldsymbol{\omega}^{S_1} \times \cancelto{0}{{}_{S_1}\mathbf{v}^{P/S_1}}$$

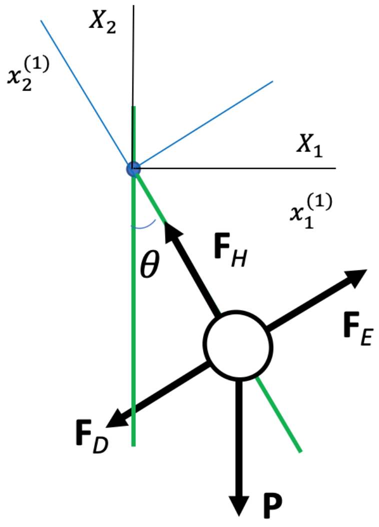
$${}^{I}_{S_1}\boldsymbol{\alpha}^{S_1} \times {}_{S_1}\mathbf{p}^{P/S_1} = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \ddot{\theta} \\ 0 & -L & 0 \end{vmatrix} = \begin{Bmatrix} L\ddot{\theta} \\ 0 \\ 0 \end{Bmatrix}$$

$${}^{I}_{S_1}\boldsymbol{\omega}^{S_1} \times \left({}^{I}_{S_1}\boldsymbol{\omega}^{S_1} \times {}_{S_1}\mathbf{p}^{P/S_1} \right) = {}^{I}_{S_1}\boldsymbol{\omega}^{S_1} \times \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \dot{\theta} \\ 0 & -L & 0 \end{vmatrix} = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \dot{\theta} \\ L\dot{\theta} & 0 & 0 \end{vmatrix} = \begin{Bmatrix} 0 \\ L\dot{\theta}^2 \\ 0 \end{Bmatrix}$$

$$\boxed{{}^{I}_{S_1}\mathbf{a}^P = \begin{Bmatrix} L\ddot{\theta} \\ L\dot{\theta}^2 \\ 0 \end{Bmatrix}}$$

Pêndulo





$$_{S_1}\mathbf{P} = \mathbf{T}_\theta \; _I\mathbf{P} = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ -mg \\ 0 \end{Bmatrix} = \begin{Bmatrix} -mg\, s\theta \\ -mg\, c\theta \\ 0 \end{Bmatrix}$$

$$_{S_1}\mathbf{F}_D = \begin{Bmatrix} -\gamma L\, \dot{\theta} \\ 0 \\ 0 \end{Bmatrix} \qquad _{S_1}\mathbf{F}_E = \begin{Bmatrix} F(t) \\ 0 \\ 0 \end{Bmatrix} \qquad _{S_1}\mathbf{F}_H = \begin{Bmatrix} 0 \\ T \\ 0 \end{Bmatrix}$$

Equações de movimento:

$$m \; _{S_1}^{I}\mathbf{a}^P = m \begin{Bmatrix} L\ddot{\theta} \\ L\dot{\theta}^2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -mg\, s\theta - \gamma L\, \dot{\theta} + F(t) \\ -mg\, c\theta + T \\ 0 \end{Bmatrix}$$

$$mL\ddot{\theta} + \gamma L\, \dot{\theta} + mg\, s\theta = F(t)$$

$$mL\dot{\theta}^2 + mg\, c\theta - T = 0$$

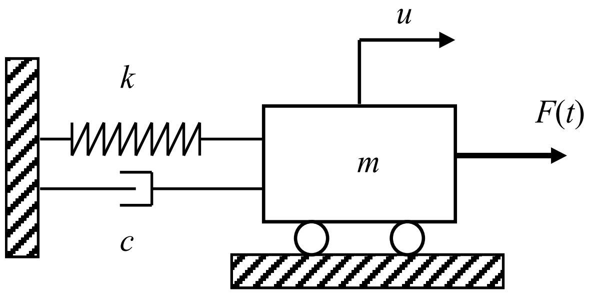
$$\boxed{\ddot{\theta} + \zeta\dot{\theta} + \omega_0^2\, s\theta = f(t)}$$

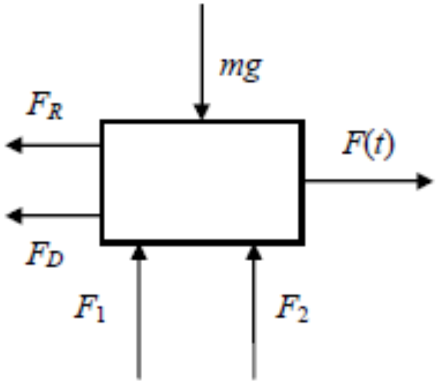
$$\zeta = \frac{\gamma}{m}$$

$$\omega_0^2 = g/L$$

$$f(t) = F(t)/mL$$

Oscilador Linear

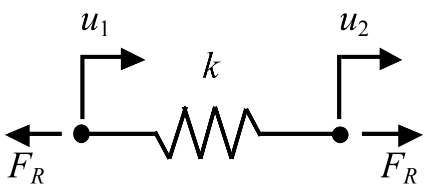


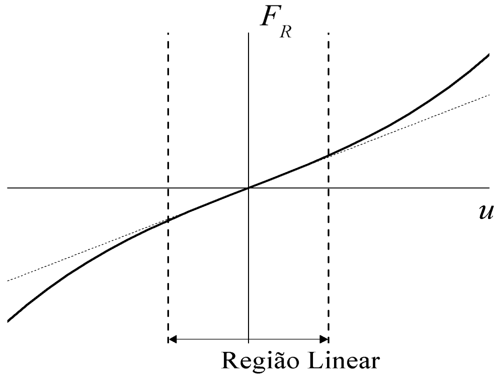


$$mg - F_1 - F_2 = 0$$

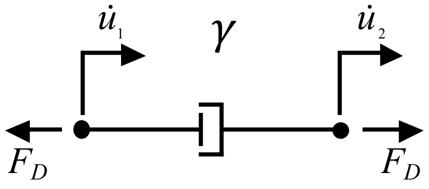
$$F(t) - F_R - F_D = m\ddot{u}$$

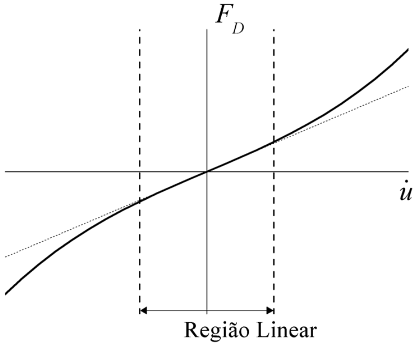
Equações constitutivas:





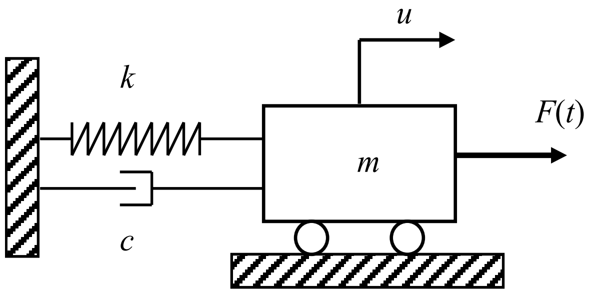
$$F_R = ku$$

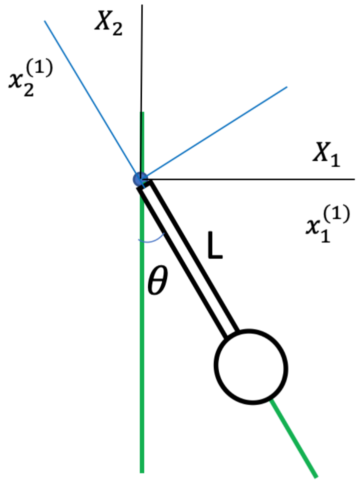




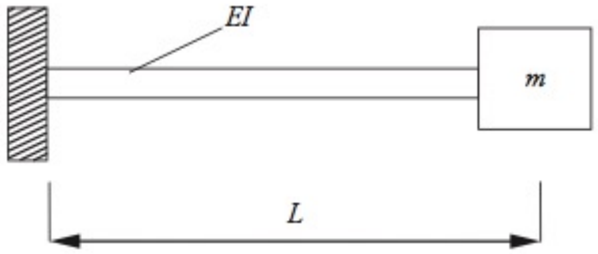
$$F_D = \gamma\dot{u}$$

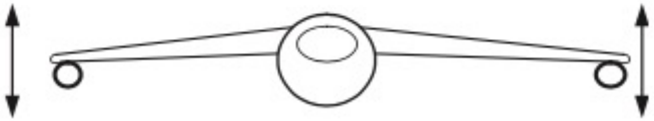
Oscilador Linear





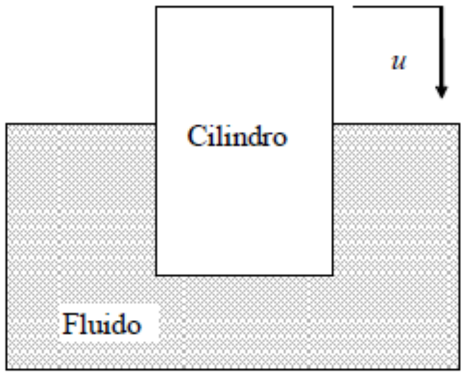
$$m\ddot{u} + \gamma\dot{u} + ku = F(t)$$

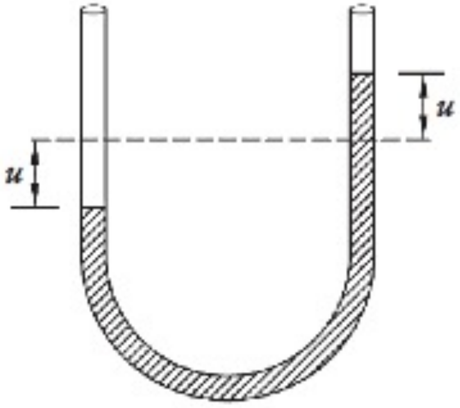




$$\boxed{\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2\, u = f(t)}$$

$$2\xi\omega_n = \frac{\gamma}{m} \qquad \omega_n^2 = \frac{k}{m} \qquad f(t) = \frac{F(t)}{m}$$





Oscilador Linear

$$\boxed{\ddot{\theta} + \zeta\dot{\theta} + \omega_0^2\,\theta = f(t)} \Rightarrow \boxed{\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2\,u = f(t)}$$

Solução da ED: $u = u_h + u_p$

Solução homogênea: $u_h = Ae^{\lambda t}$ $\quad (\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2)Ae^{\lambda t} = 0$

$$\boxed{\lambda_{1,2} = \omega_n\left(-\xi \pm \sqrt{\xi^2 - 1}\right)}$$

$$\boxed{\lambda = Re(\lambda) + i\,Im(\lambda)}$$

$\xi = 0:$ $\quad \lambda_{1,2} = \pm i\omega_n \quad \rightarrow$ Vibrações harmônicas

$0 < \xi < 1:$ $\quad \lambda_{1,2} = \omega_n\left(-\xi \pm i\sqrt{1 - \xi^2}\right) \quad \rightarrow$ Vibrações subamortecidas

$\xi = 1:$ $\quad \lambda_{1,2} = -\omega_n \quad \rightarrow$ Movimento criticamente amortecido

$\xi > 1:$ $\quad \lambda_{1,2} = \omega_n\left(-\xi \pm \sqrt{\xi^2 - 1}\right) \quad \rightarrow$ Movimento superamortecido

$\xi < 0:$ $\quad \lambda_{1,2} = \omega_n\left(+\xi \pm \sqrt{\xi^2 - 1}\right) \quad \rightarrow$ Movimento instável

Vibrações Livres Harmônicas

$$\ddot{u} - \omega_n^2 u = 0 \quad \Rightarrow \quad \begin{gathered} \xi = 0 \\ \lambda_{1,2} = \pm i\omega_n \end{gathered}$$

A solução é uma combinação linear das duas soluções:

$$u = A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t}$$

Usando a Identidade de Euler: $e^{i\theta} = c\theta + i\, s\theta$

$$u = C_1 c(\omega_n t) + C_2 s(\omega_n t)$$

onde: $C_1 = A_1 - A_2$

$C_2 = i(A_1 - A_2)$

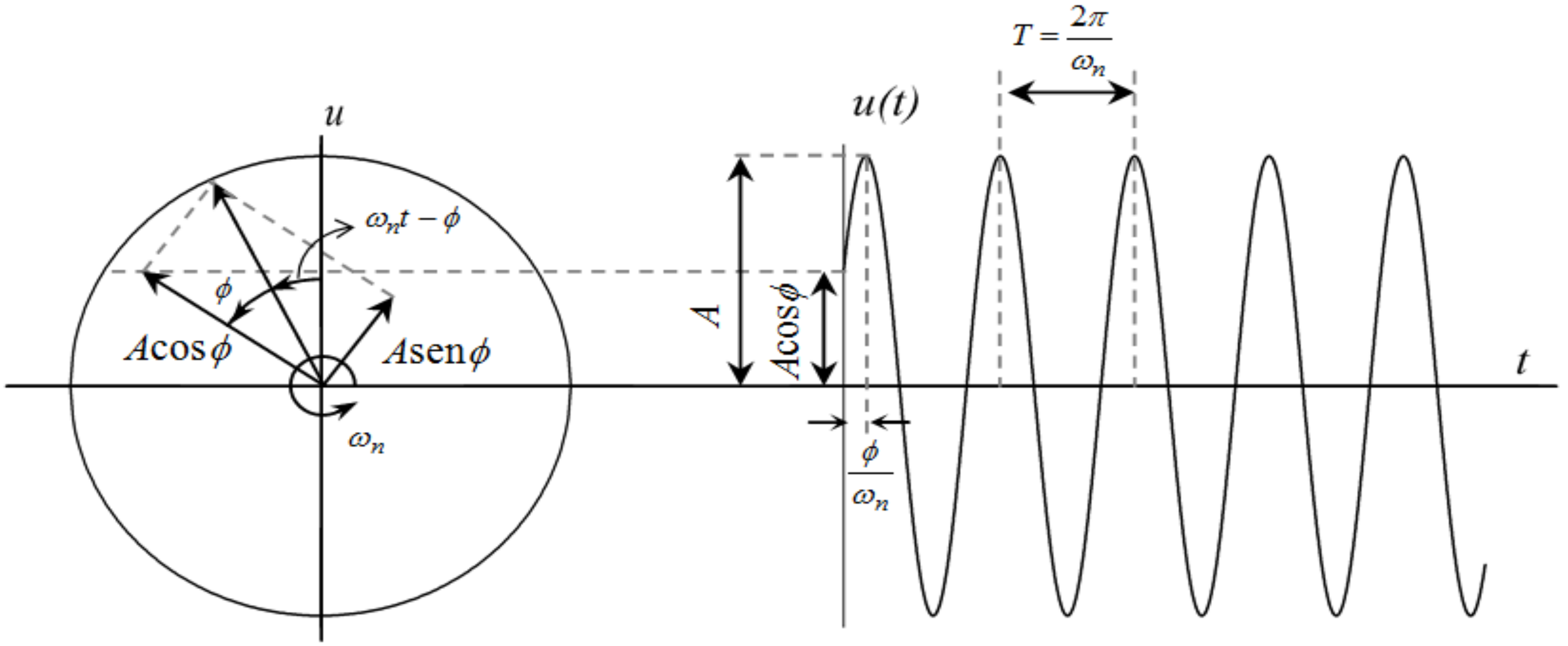
Vibrações Livres Harmônicas

Fazendo:

$$C_1 = A_1 + A_2 = A\,c\phi$$
$$C_2 = i(A_1 - A_2) = A\,s\phi$$

Reescreve-se a solução da seguinte forma:

$$\boxed{u = A\,c(\omega_n t - \phi)}$$



Vibrações Livres Harmônicas

As condições iniciais definem as constantes de integração, ou os pesos da combinação linear.

$$\begin{cases} u(0) = u_0 \\ \dot{u}(0) = v_0 \end{cases}$$

Com isso: $$\begin{cases} u(0) = C_1 = u_0 \\ \dot{u}(0) = C_2\omega_n = v_0 \end{cases}$$

$$A\, c\phi = u_0$$

$$A\, s\phi = v_0/\omega_n$$

$$A = \sqrt{u_0^2 + \frac{v_0^2}{\omega_n^2}} \qquad \mathrm{tg}(\varphi) = \frac{v_0}{u_0\omega_n}$$

Vibrações Livres Harmônicas: Espaço de Estado

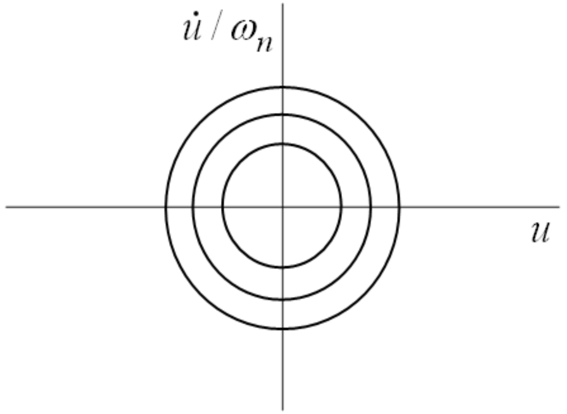
$$u = u_0\, c(\omega_n t) + \frac{v_0}{\omega_n}\, s(\omega_n t)$$

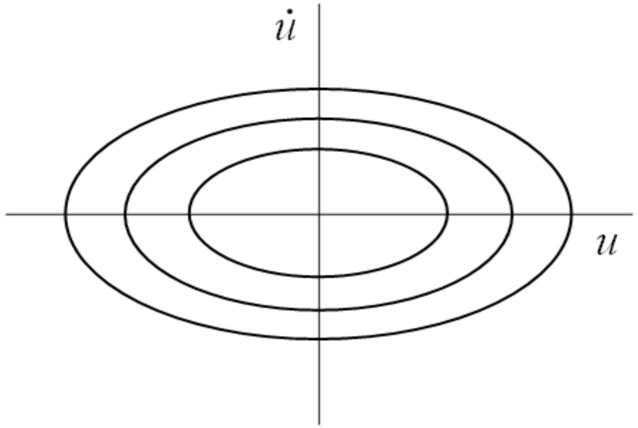
$$\frac{\dot{u}}{\omega_n} = -u_0\, s(\omega_n t) + \frac{v_0}{\omega_n}\, c(\omega_n t)$$

Elevando ao quadrado e somando:

$$u^2 + \left(\frac{\dot{u}}{\omega_n}\right)^2 = A^2$$

Equação da circunferência



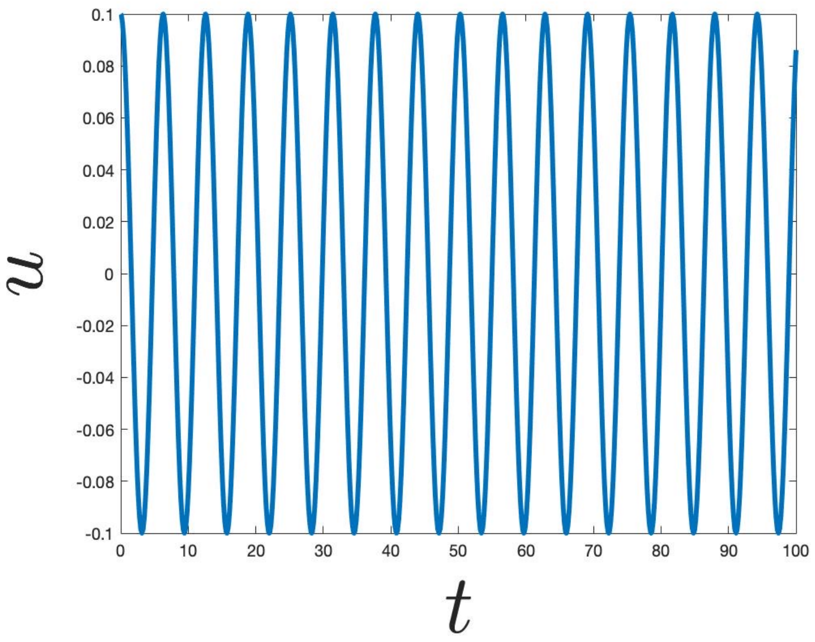


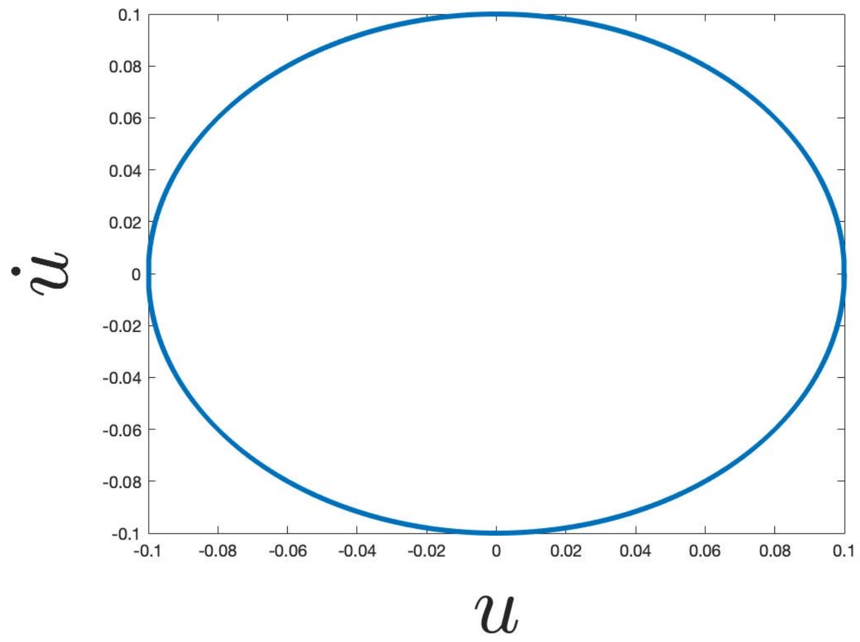
Forma canônica: $\dot{x} = f(x,t), x \in R^n$

$$\dot{u} = v$$
$$\dot{v} = f(t) - \omega_n^2\, u - 2\xi\omega_n v$$

Vibrações Livres Harmônicas

$$\boxed{u = A\,c(\omega_n t - \phi)}$$





Vibrações Livres Subamortecidas

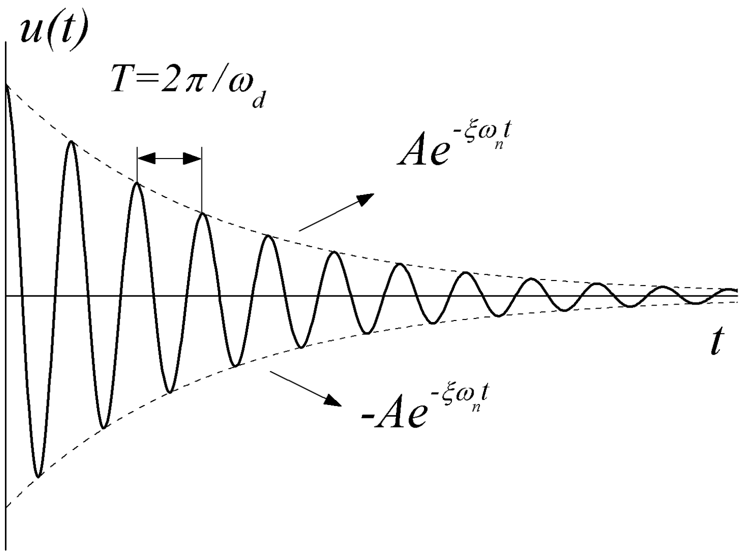
$$\ddot{u} + 2\xi\omega_n \dot{u} + \omega_n^2 u = 0 \quad \Rightarrow \quad \begin{gathered} 0 < \xi < 1 \\ \lambda_{1,2} = \omega_n\left(-\xi \pm i\sqrt{1-\xi^2}\right) \end{gathered}$$

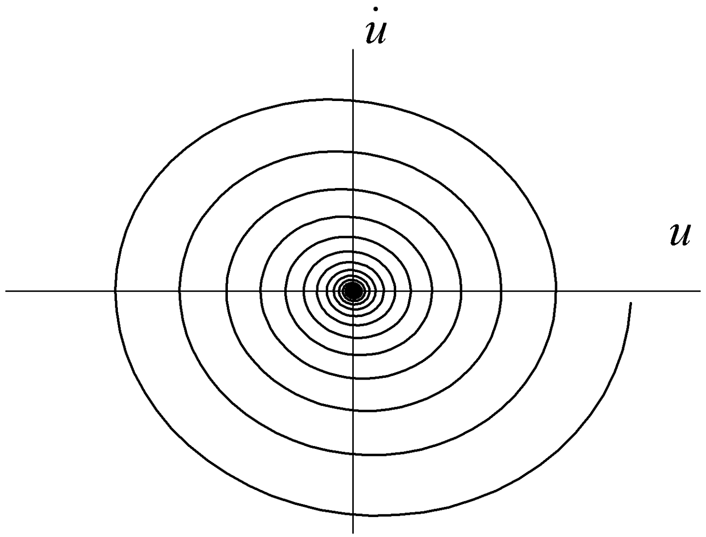
$$\begin{aligned} u_h &= e^{-\xi\omega_n t}\left[C_1\, c(\omega_d t) + C_2\, s(\omega_d t)\right] \\ &= C_A\, e^{-\xi\omega_n t}\, c(\omega_d t - \phi) \end{aligned}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

Condições iniciais:

$$\begin{cases} A\cos(\phi) = C_1 = u_0 \\ A\sin(\phi) = C_2 = \dfrac{v_0 + u_0\xi\omega_n}{\omega_d} \end{cases}$$





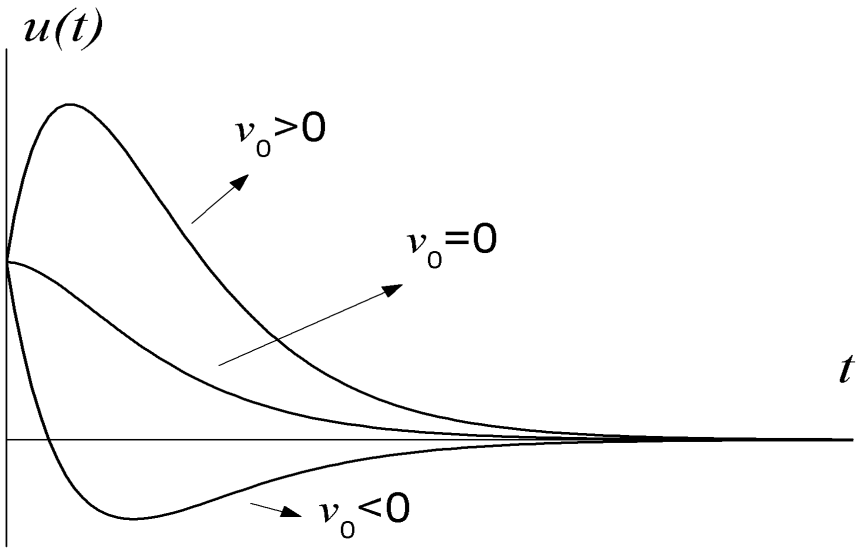
Movimento Criticamente Amortecido

$$\ddot{u} + 2\xi\omega_n \dot{u} + \omega_n^2 u = 0 \quad \Rightarrow \quad \xi = 1, \quad \lambda_{1,2} = -\omega_n$$

Solução: $u = e^{-\omega_n t}\left(A_1 + A_2 t\right)$

Condições iniciais: $\begin{cases} u(0) = A_1 = u_0 \\ \dot{u}(0) = -A_1\omega_n + A_2 = v_0 \end{cases} \Rightarrow \quad A_2 = v_0 + u_0\omega_n$

$$\boxed{u = e^{-\omega_n t}\left[u_0 + \left(v_0 + u_0\omega_n\right)t\right]}$$



Movimento Superamortecido

$$\ddot{u} + 2\xi\omega_n \dot{u} + \omega_n^2 u = 0 \quad \Rightarrow \quad \xi > 1$$

$$\lambda_{1,2} = \omega_n \left(-\xi \pm \sqrt{\xi^2 - 1}\right)$$

Solução:

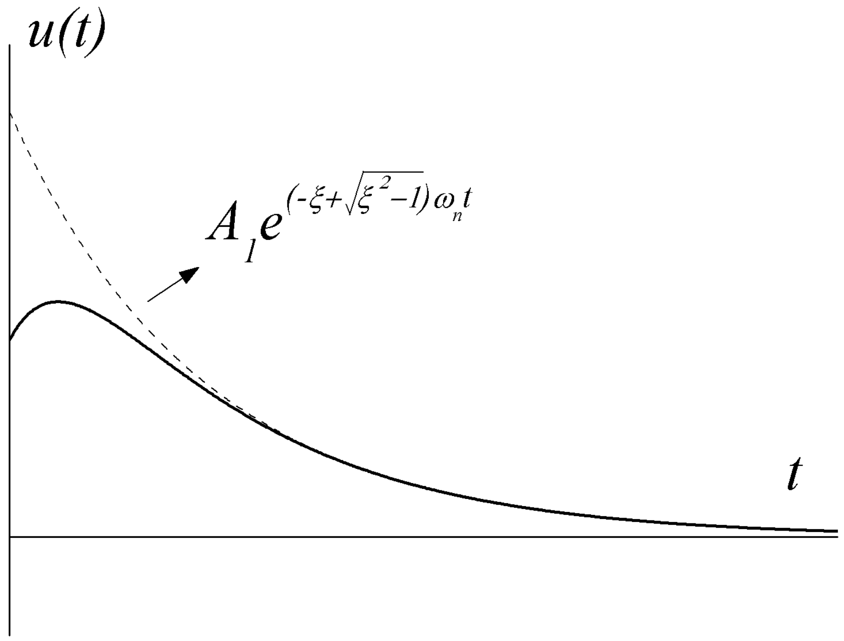
$$u = e^{-\xi\omega_n t}\left[A_1 e^{\left(\omega_n\sqrt{\xi^2-1}\right)t} + A_2 e^{-\left(\omega_n\sqrt{\xi^2-1}\right)t}\right]$$

Usando:

$$e^{\pm\theta} = \cosh(\theta) \pm \text{senh}(\theta)$$

e condições iniciais:

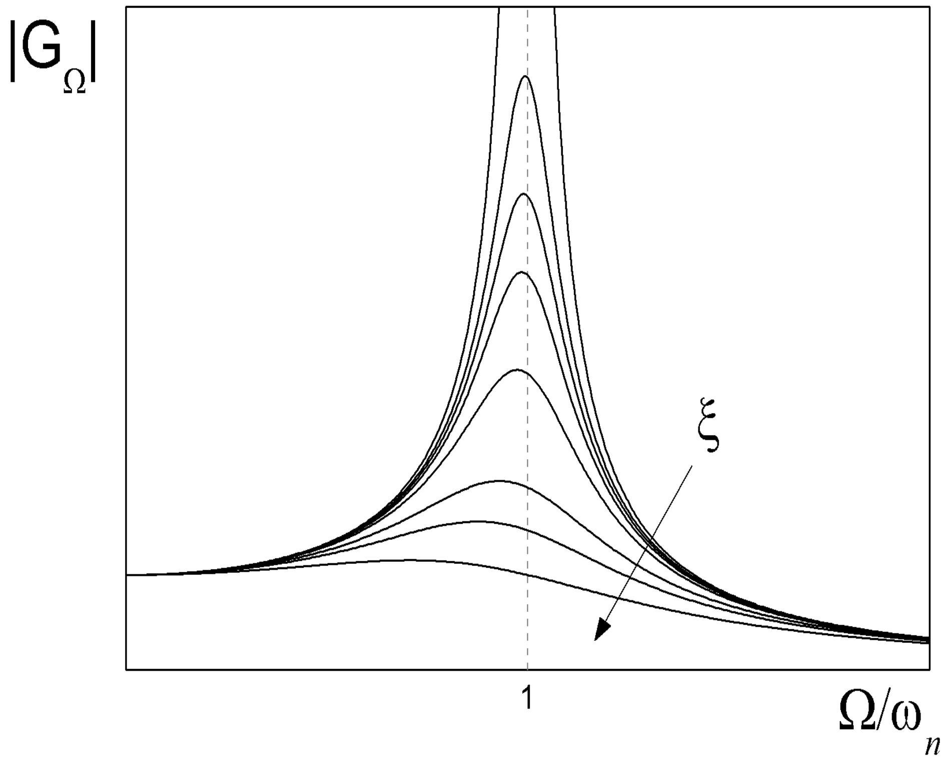
$$\begin{cases} u(0) = C_1 = u_0 \\ \dot{u}(0) = -C_1\xi\omega_n + C_2\omega_n\sqrt{\xi^2-1} = v_0 \end{cases} \Rightarrow \quad C_2 = \frac{v_0 + u_0\xi\omega_n}{\omega_n\sqrt{\xi^2-1}}$$



$$u = e^{-\xi\omega_n t}\left[u_0 \cosh(\omega_n\sqrt{\xi^2-1}\,t) + \frac{v_0 + u_0\xi\omega_n}{\omega_n\sqrt{\xi^2-1}}\text{senh}(\omega_n\sqrt{\xi^2-1}\,t)\right]$$

Oscilador Linear: Forçamento Harmônico

$$\boxed{\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2\, u = f_0 \cos(\Omega t)}$$

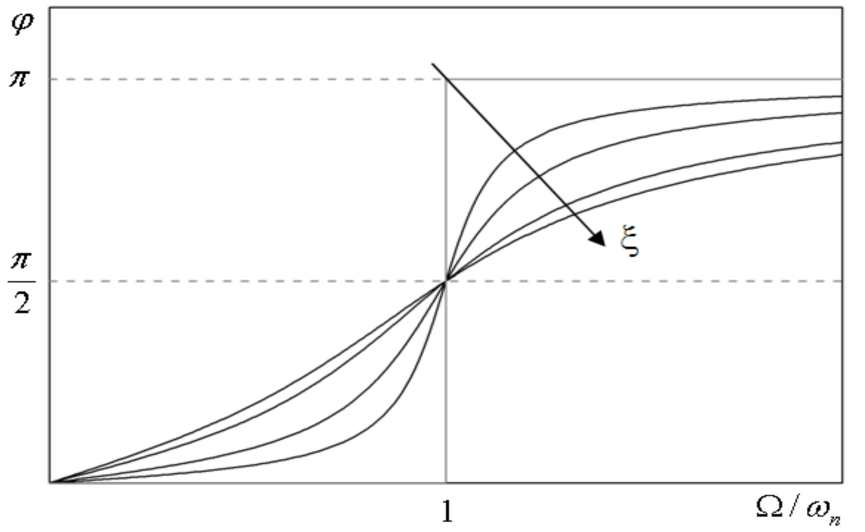


Solução particular: $u_p = K_1 s(\Omega t) + K_2\, c(\Omega t)$

$$(\omega_n^2 - \Omega^2)K_1 - 2\xi\Omega K_2 = 0$$

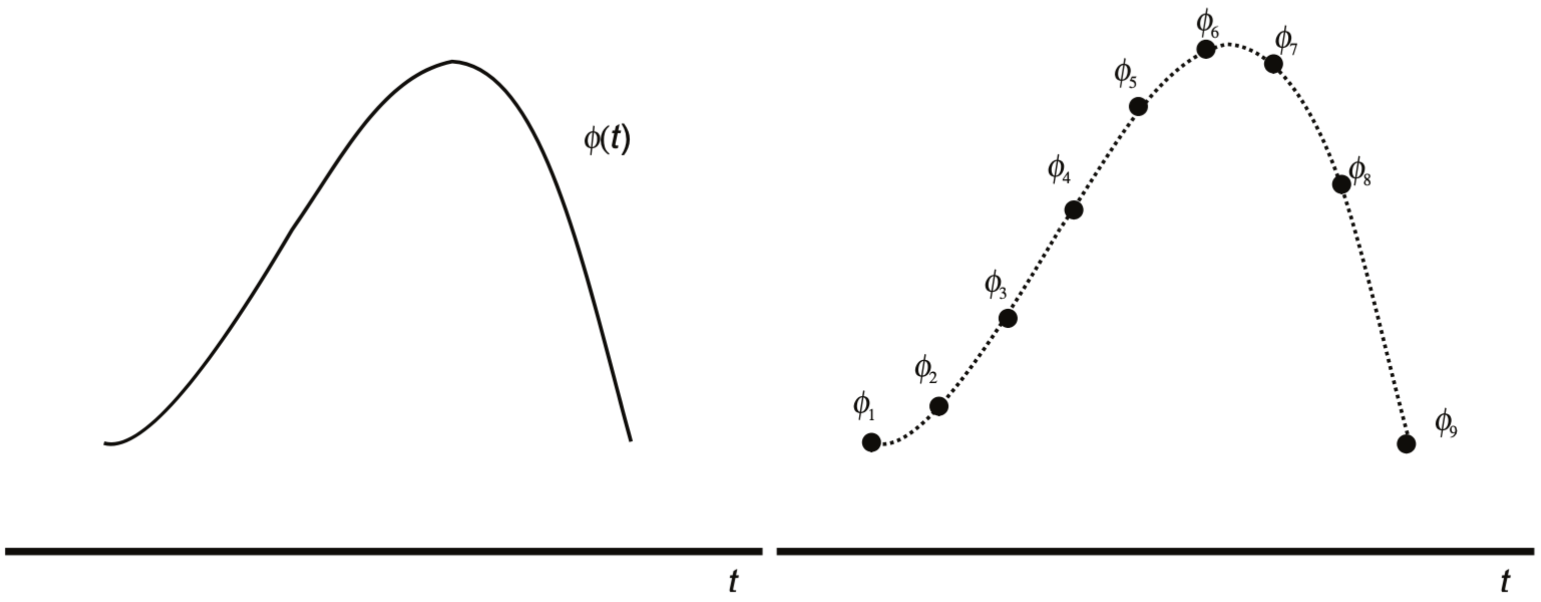
$$(\omega_n^2 - \Omega^2)K_2 - 2\xi\Omega K_1 = f_0$$

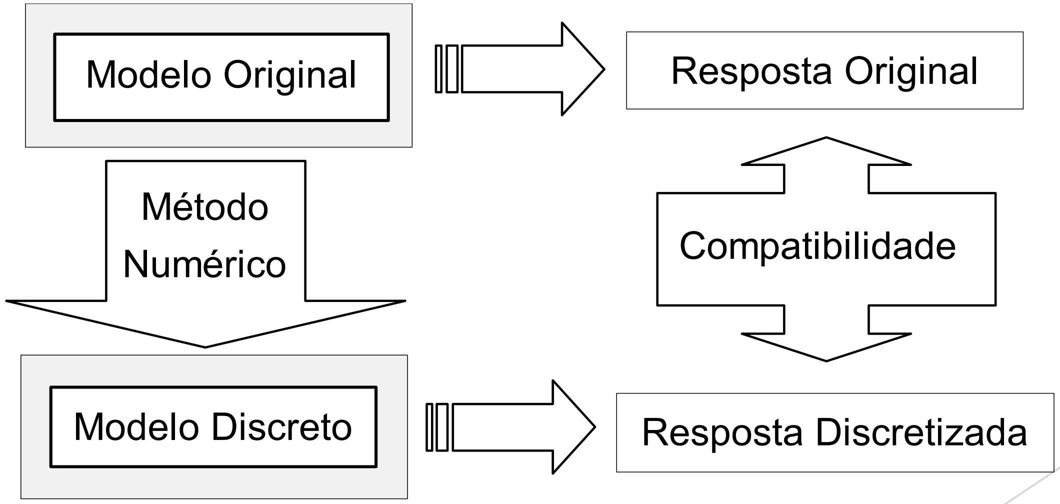
$$K_1 = \frac{2\xi\Omega}{(\omega_n^2 - \Omega^2)^2 + (2\xi\omega_n\Omega)^2} f_0 \qquad K_2 = \frac{(\omega_n^2 - \Omega^2)}{(\omega_n^2 - \Omega^2)^2 + (2\xi\omega_n\Omega)^2} f_0$$



$$\boxed{u_p = U(\Omega)\, c(\Omega t - \phi)}$$

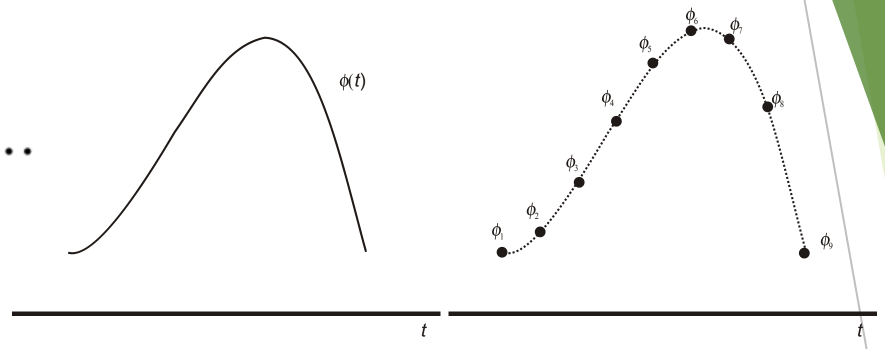
Solução Numérica





Método de Runge-Kutta

$$x(t_{i+1}) = x(t_i) + \dot{x}(t_i)\Delta t + \ddot{x}(t_i)\frac{\Delta t^2}{2!} + \dddot{x}(t_i)\frac{\Delta t^3}{3!} + \cdots$$



$$x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = \Delta t\, f(x_n, t_n)$$

$$k_2 = \Delta t\, f(x_n + \frac{k_1}{2}, t_n + \frac{\Delta t}{2})$$

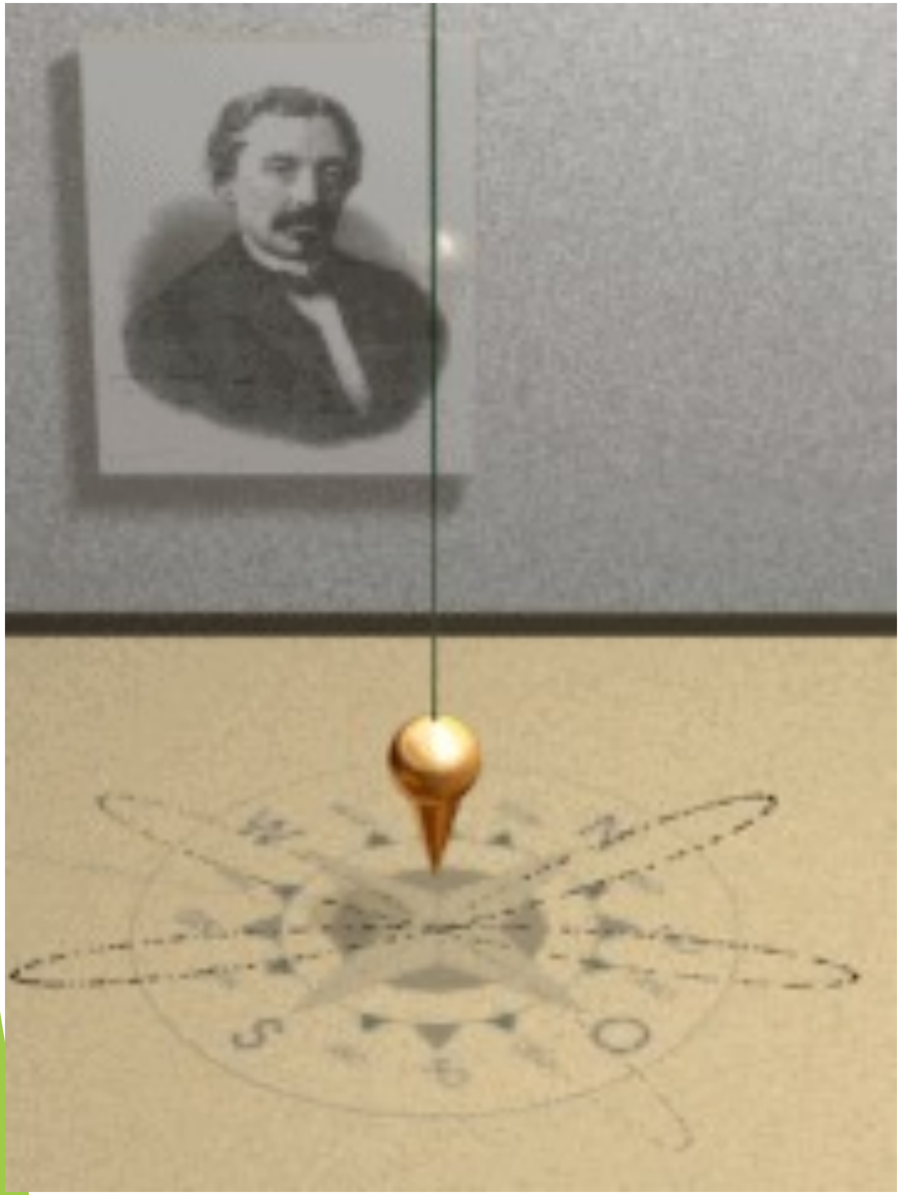
$$k_3 = \Delta t\, f(x_n + \frac{k_2}{2}, t_n + \frac{\Delta t}{2})$$

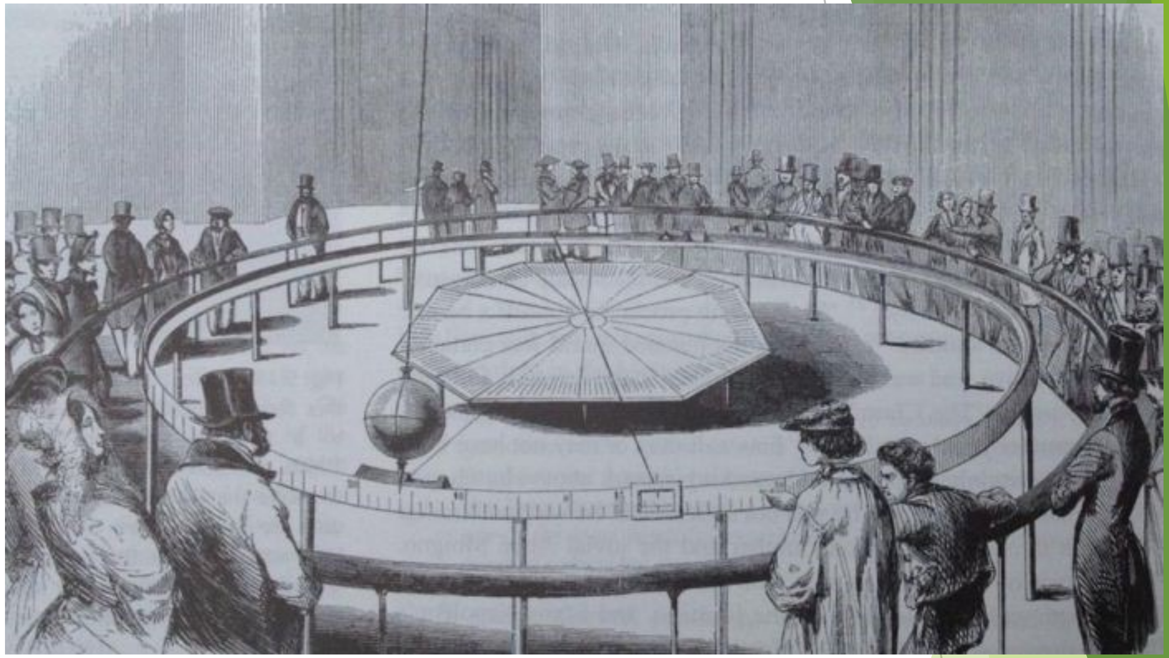
$$k_4 = \Delta t\, f(x_n + k_3, t_n + \Delta t)$$

Caixa 2.1: Runge-Kutta de quarta ordem.

```
RK4
{
hh=h*0.5;
h6=h/6.0;
th=t+hh;
(*derivs)(t,x,dxdt);
for (i=0;i<=n-1;i++) xt[i]=x[i]+hh*dxdt[i];
(*derivs)(th,xt,dxt);
for (i=0;i<=n-1;i++) xt[i]=x[i]+hh*dxt[i];
(*derivs)(th,xt,dxm);
for (i=0;i<=n-1;i++) {
    xt[i]=x[i]+h*dxm[i];
    dxm[i] += dxt[i];
}
(*derivs)(t+h,xt,dxt);
for (i=0;i<=n-1;i++) {
    xout[i]=x[i]+h6*(dxdt[i]+dxt[i]+2.0*dxm[i]);
}
}
```

Pêndulo de Foucault

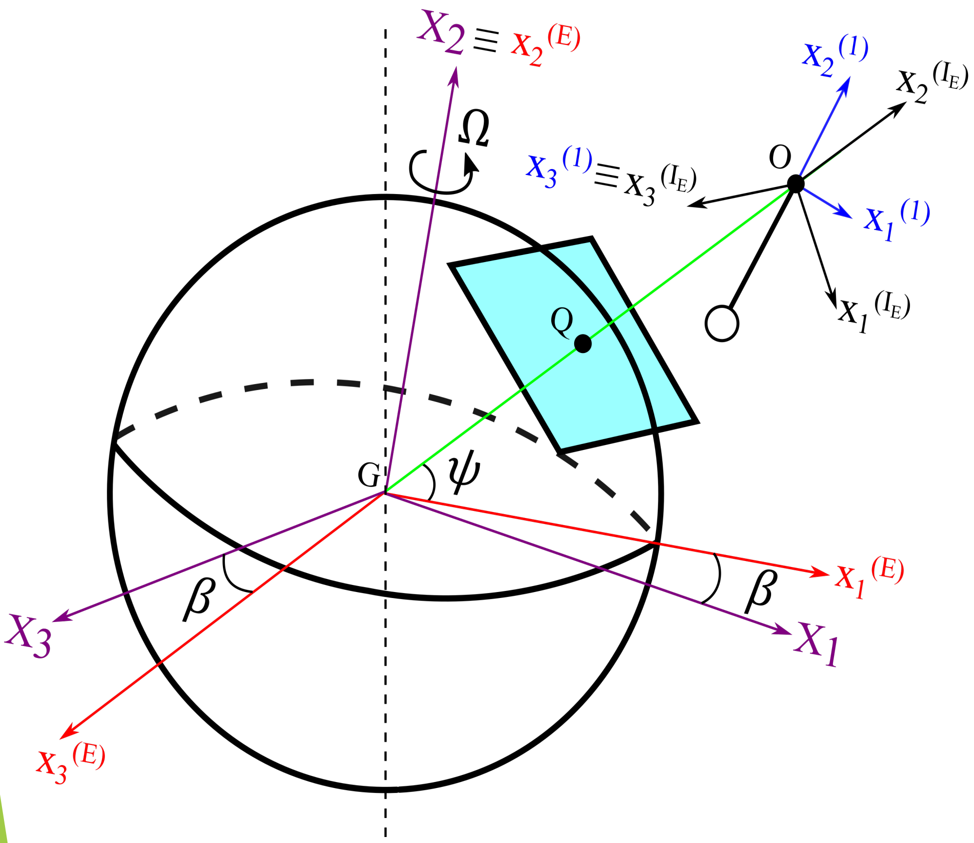


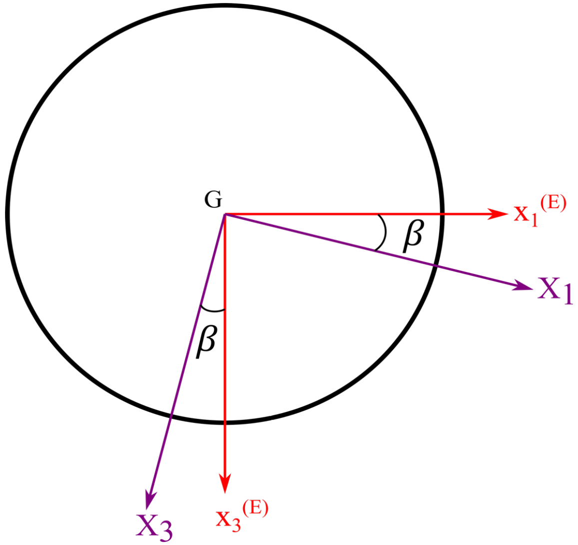


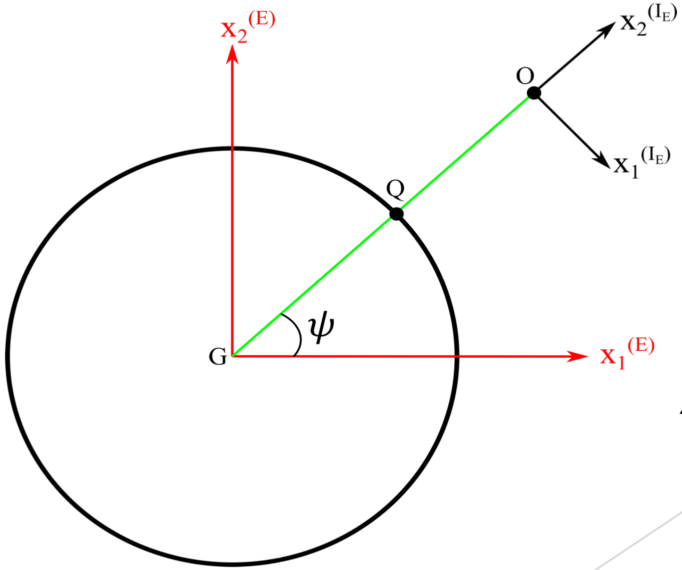
Michel Foucault (1819-1868) – experiência realizada no Panthéon (Paris) em 1851 foi a primeira prova de que a Terra gira.

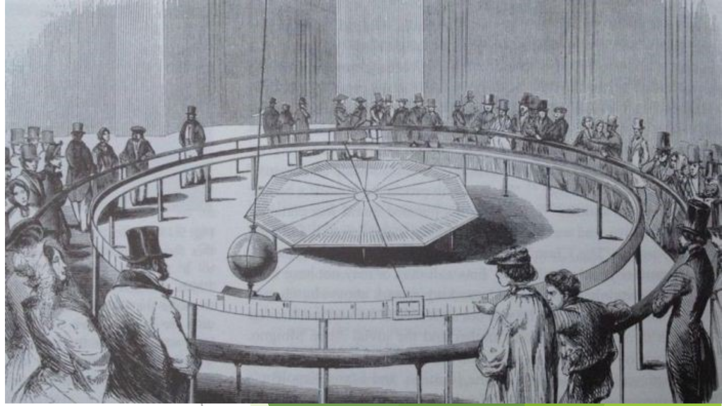
Para analisar o pêndulo de Foucault deve-se considerar que o referencial *I* tratado no caso anterior, de fato gira, pois está solidário à Terra.

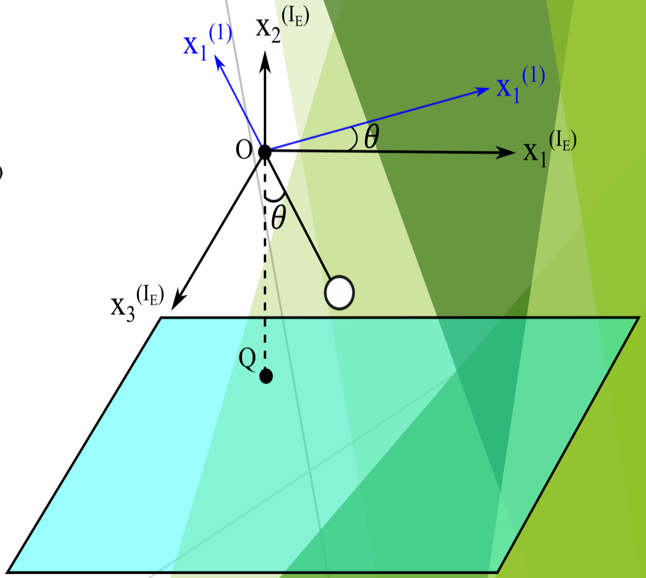
Pêndulo de Foucault

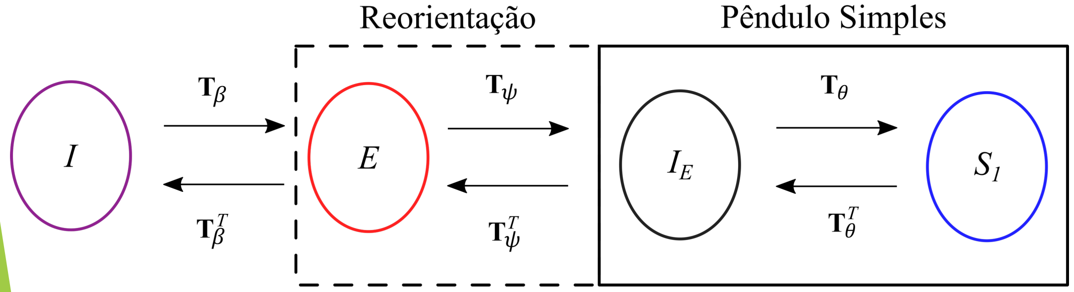




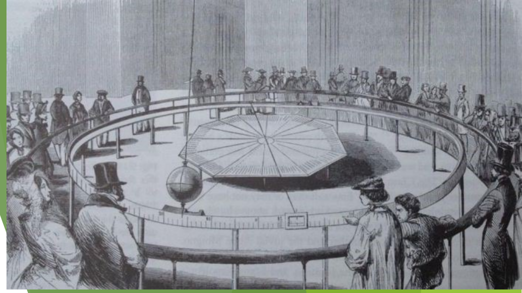


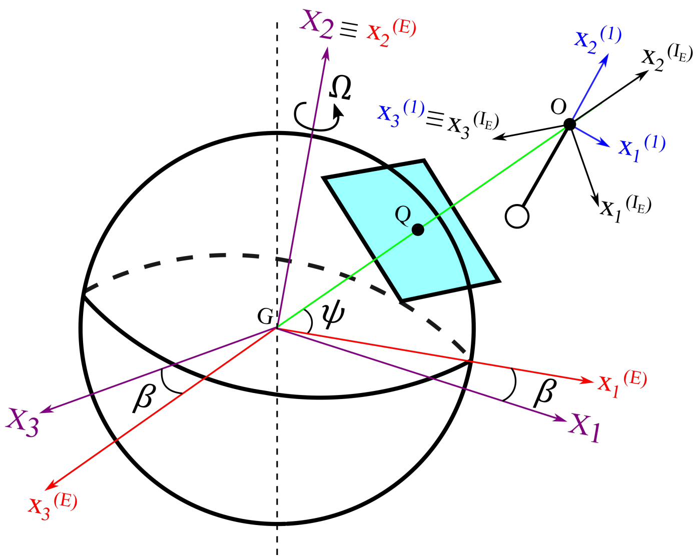


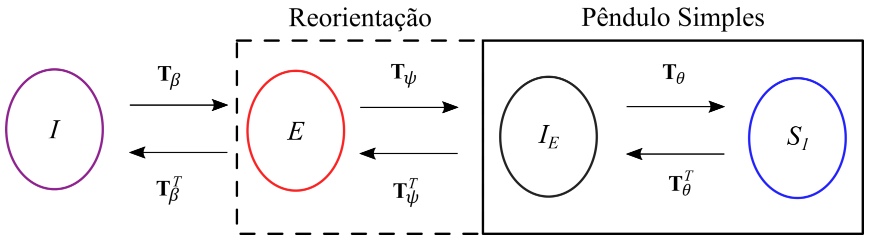




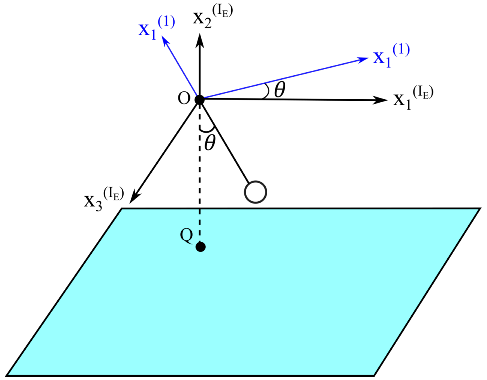
Pêndulo de Foucault







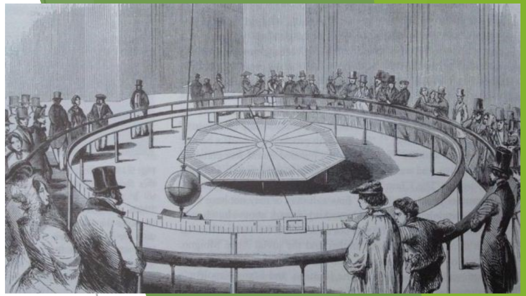
$$\mathbf{T}_\beta = \begin{bmatrix} c\beta & 0 & -s\beta \\ 0 & 1 & 0 \\ s\beta & 0 & c\beta \end{bmatrix} \qquad \dot{\beta} = \Omega$$

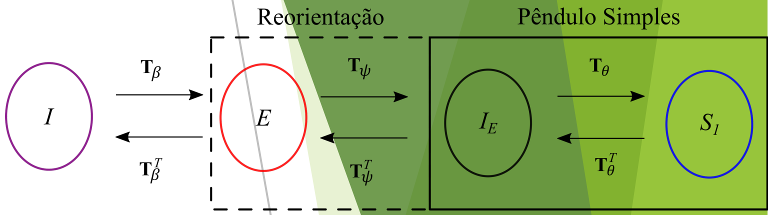


$$\mathbf{T}_\psi = \begin{bmatrix} c(\pi/2-\psi) & s(\pi/2-\psi) & 0 \\ -s(\pi/2-\psi) & c(\pi/2-\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s\psi & c\psi & 0 \\ -c\psi & s\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_\theta = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pêndulo de Foucault: Velocidade Angular





$$ {}_{S_1}^{I}\boldsymbol{\omega}^{S_1} = {}_{S_1}^{I}\boldsymbol{\omega}^{E} + {}_{S_1}^{E}\boldsymbol{\omega}^{I_E} + {}_{S_1}^{I_E}\boldsymbol{\omega}^{S_1} $$

$$ {}_{I}^{I}\boldsymbol{\omega}^{E} = \begin{Bmatrix} 0 \\ \dot{\beta} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \Omega \\ 0 \end{Bmatrix} $$

$$ {}_{E}^{E}\boldsymbol{\omega}^{I_E} = \begin{Bmatrix} 0 \\ 0 \\ -\dot{\psi} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} $$

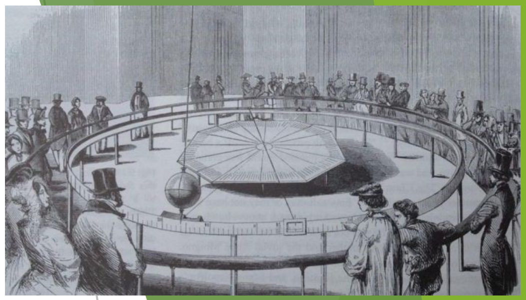
$$ {}_{S_0}^{I_E}\boldsymbol{\omega}^{S_1} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} $$

$$ {}_{S_1}^{I}\boldsymbol{\omega}^{S_1} = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s\psi & c\psi & 0 \\ -c\psi & s\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & -s\beta \\ 0 & 1 & 0 \\ s\beta & 0 & c\beta \end{bmatrix} \begin{Bmatrix} 0 \\ \Omega \\ 0 \end{Bmatrix} + \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} $$

$$ {}_{S_1}^{I}\boldsymbol{\omega}^{S_1} = \begin{Bmatrix} \Omega\,(c\psi c\theta + s\psi s\theta) \\ \Omega\,(s\psi c\theta - c\psi s\theta) \\ \dot{\theta} \end{Bmatrix} $$

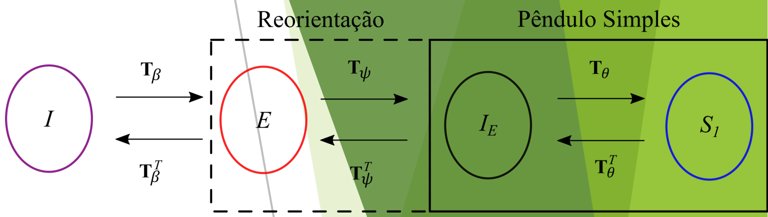
$$ {}_{S_1}^{I}\boldsymbol{\omega}^{I_E} = {}_{S_1}^{I}\boldsymbol{\omega}^{E} = \begin{Bmatrix} \Omega\,(c\psi c\theta + s\psi s\theta) \\ \Omega\,(s\psi c\theta - c\psi s\theta) \\ 0 \end{Bmatrix} $$

Pêndulo de Foucault: Aceleração Angular



$$ {}^{I}_{S_1}\boldsymbol{\alpha}^{S_1} = {}^{I_E}_{S_1}\boldsymbol{\alpha}^{S_1} + {}^{I}_{S_1}\boldsymbol{\omega}^{I_E} \times {}^{I_E}_{S_1}\boldsymbol{\omega}^{S_1} $$

$$ \ddot{\beta} = \dot{\Omega} = \ddot{\psi} = \ddot{\phi} = 0 $$

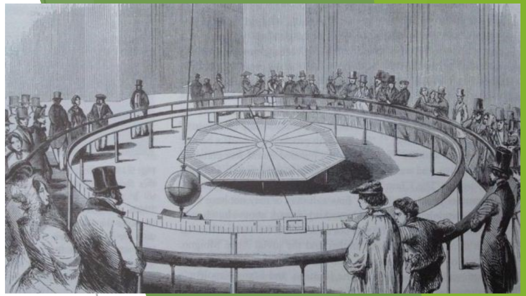


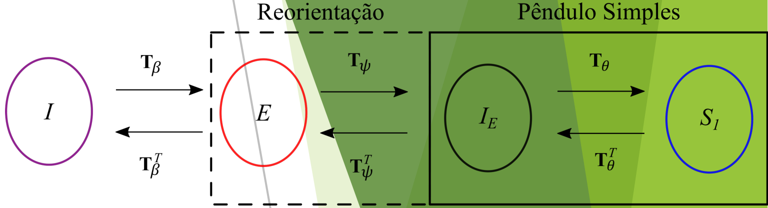
$$ {}^{I_E}_{S_1}\boldsymbol{\alpha}^{S_1} = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \ddot{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \ddot{\theta} \end{Bmatrix} $$

$$ {}^{I}_{S_1}\boldsymbol{\omega}^{I_E} \times {}^{I_E}_{S_1}\boldsymbol{\omega}^{S_1} = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ \Omega\,(c\psi c\theta + s\psi s\theta) & \Omega\,(s\psi c\theta - c\psi s\theta) & 0 \\ 0 & 0 & \dot{\theta} \end{vmatrix} = \begin{Bmatrix} \Omega\dot{\theta}\,(s\psi c\theta - c\psi s\theta) \\ -\Omega\dot{\theta}\,\,(c\psi c\theta + s\psi s\theta) \\ 0 \end{Bmatrix} $$

$$ {}^{I}_{S_1}\boldsymbol{\alpha}^{I_E} = 0 $$

Pêndulo de Foucault: Aceleração





$${}_{S_1}^{I}\mathbf{a}^P = {}_{S_1}^{I}\mathbf{a}^O + \boxed{{}_{S_1}^{I_E}\mathbf{a}^P} + {}_{S_1}^{I}\boldsymbol{\omega}^{I_E}\times\left({}_{S_1}^{I}\boldsymbol{\omega}^{I_E}\times{}_{S_1}^{I_E}\mathbf{p}^P\right) + {}_{S_1}^{I}\boldsymbol{\alpha}^{I_E}\times{}_{S_1}^{I_E}\mathbf{p}^P + 2\,{}_{S_1}^{I}\boldsymbol{\omega}^{I_E}\times{}_{S_1}^{I_E}\mathbf{v}^P$$

$${}_{S_1}^{I_E}\mathbf{a}^P = {}_{S_1}\mathbf{a}^{P/S_1} = \begin{Bmatrix} L\ddot{\theta} \\ L\dot{\theta}^2 \\ 0 \end{Bmatrix}$$

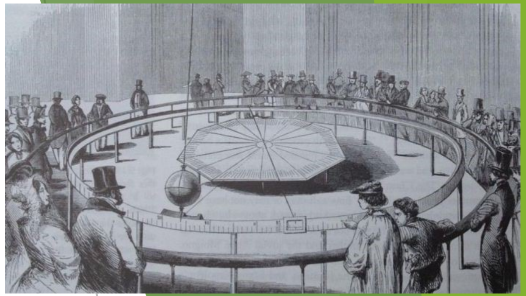
$${}_{S_1}^{I}\boldsymbol{\omega}^{I_E}\times\left({}_{S_1}^{I}\boldsymbol{\omega}^{I_E}\times{}_{S_1}\mathbf{p}^{P/I_E}\right) = {}_{S_1}^{I}\boldsymbol{\omega}^{I_E}\times\begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ \Omega\,(c\psi c\theta + s\psi s\theta) & \Omega\,(s\psi c\theta - c\psi s\theta) & 0 \\ 0 & -L & 0 \end{vmatrix} =$$

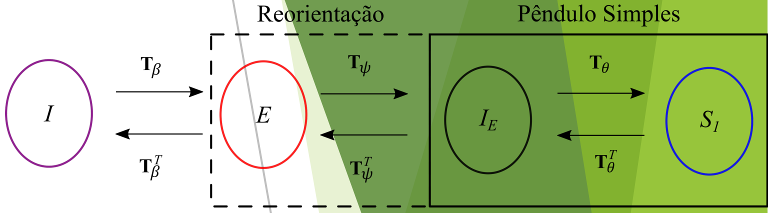
$$\begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ \Omega\,(c\psi c\theta + s\psi s\theta) & \Omega\,(s\psi c\theta - c\psi s\theta) & 0 \\ 0 & 0 & -L\Omega\,(c\psi c\theta + s\psi s\theta) \end{vmatrix} = \begin{Bmatrix} -L\Omega^2\,(s\psi c\theta - c\psi s\theta)(c\psi c\theta + s\psi s\theta) \\ L\Omega^2\,(c\psi c\theta + s\psi s\theta)(c\psi c\theta + s\psi s\theta) \\ 0 \end{Bmatrix}$$

$$2\,{}_{S_1}^{I}\boldsymbol{\omega}^{I_E}\times{}_{S_1}\mathbf{v}^{P/I_E} = 2\begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ \Omega\,(c\psi c\theta + s\psi s\theta) & \Omega\,(s\psi c\theta - c\psi s\theta) & 0 \\ L\dot{\theta} & 0 & 0 \end{vmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ -2L\dot{\theta}\Omega\,(s\psi c\theta - c\psi s\theta) \end{Bmatrix}$$

Pêndulo de Foucault: Aceleração

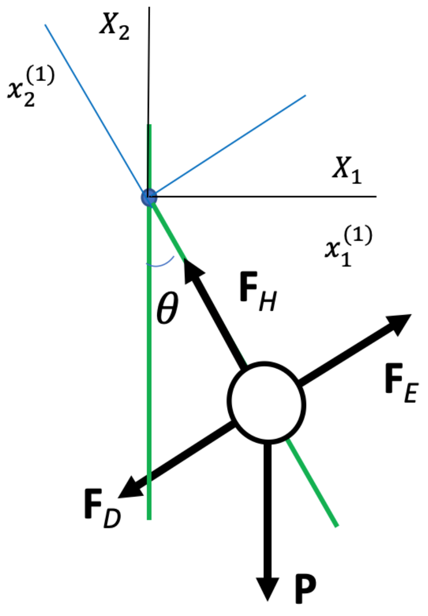




$$
{}^{I}_{S_1}\mathbf{a}^P = {}^{I}_{S_1}\mathbf{a}^O + {}^{I_E}_{S_1}\mathbf{a}^P + {}^{I}_{S_1}\boldsymbol{\omega}^{I_E}\times\left({}^{I}_{S_1}\boldsymbol{\omega}^{I_E}\times{}^{I_E}_{S_1}\mathbf{p}^P\right) + {}^{I}_{S_1}\boldsymbol{\alpha}^{I_E}\times{}^{I_E}_{S_1}\mathbf{p}^P + 2\,{}^{I}_{S_1}\boldsymbol{\omega}^{I_E}\times{}^{I_E}_{S_1}\mathbf{v}^P
$$

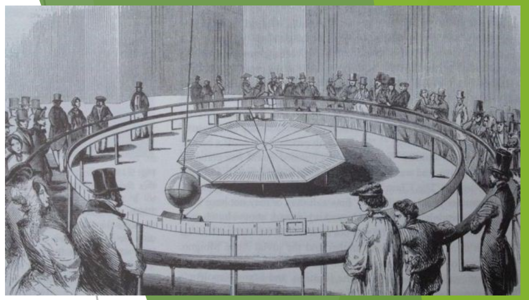
$$
\boxed{{}^{I}_{S_1}\mathbf{a}^P = \begin{Bmatrix} L\ddot{\theta} - L\Omega^2\,(s\psi c\theta - c\psi s\theta)(c\psi c\theta + s\psi s\theta) \\ L\dot{\theta}^2 + L\Omega^2\,(c\psi c\theta + s\psi s\theta)(c\psi c\theta + s\psi s\theta) \\ -2L\dot{\theta}\Omega\,(s\psi c\theta - c\psi s\theta) \end{Bmatrix}}
$$

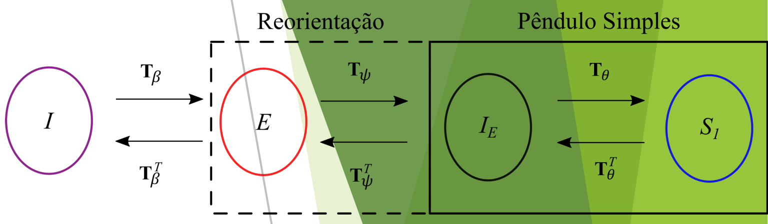
Pêndulo de Foucault: Forças



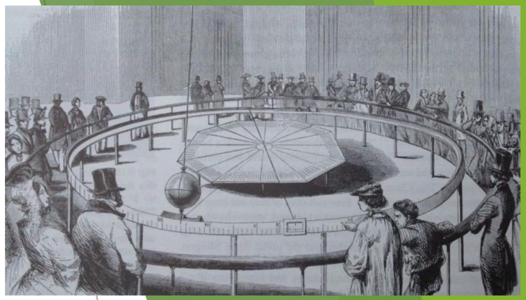
$$_{s_1}\mathbf{F}^P = {}_{s_1}\mathbf{F}^P + \mathbf{T}_\theta\, {}_{s_0}\mathbf{P} = \begin{Bmatrix} 0 \\ T \\ 0 \end{Bmatrix} + \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ -mg \\ 0 \end{Bmatrix}$$

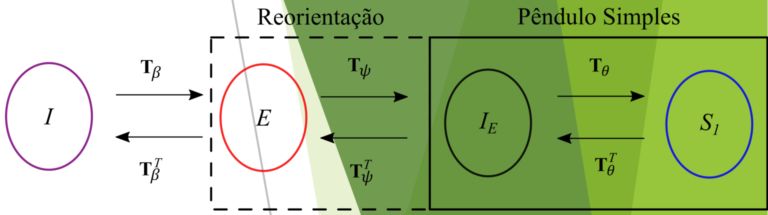
$$\boxed{_{s_1}\mathbf{F}^P = \begin{Bmatrix} -mg\, s\theta \\ T - mg\, c\theta \\ 0 \end{Bmatrix}}$$





Pêndulo de Foucault: Equações de Movimento



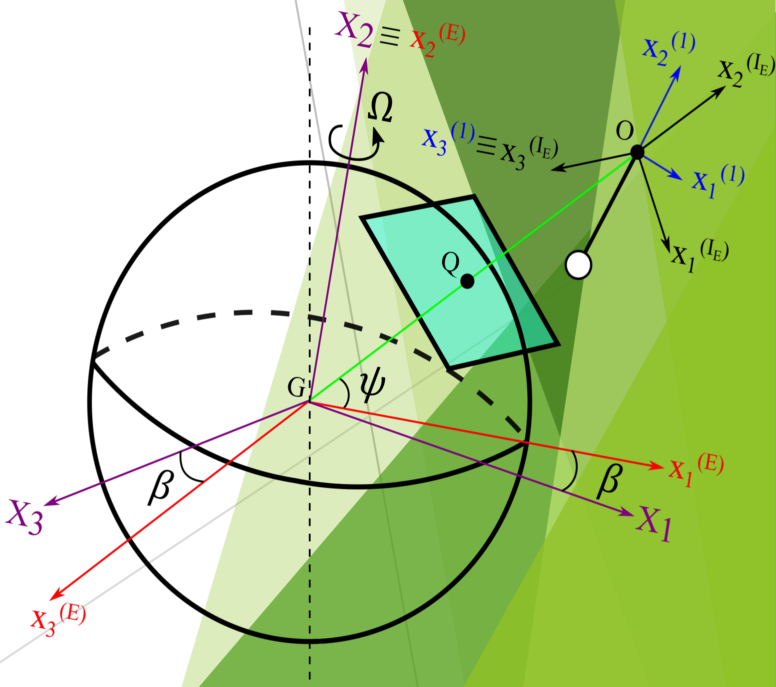


$$\begin{Bmatrix} -mg\, s\theta \\ T - mg\, c\theta \\ 0 \end{Bmatrix} = m \begin{Bmatrix} L\ddot{\theta} - L\Omega^2\,(s\psi c\theta - c\psi s\theta)(c\psi c\theta + s\psi s\theta) \\ L\dot{\theta}^2 + L\Omega^2\,(c\psi c\theta + s\psi s\theta)(c\psi c\theta + s\psi s\theta) \\ -2L\dot{\theta}\Omega\,(s\psi c\theta - c\psi s\theta) \end{Bmatrix}$$

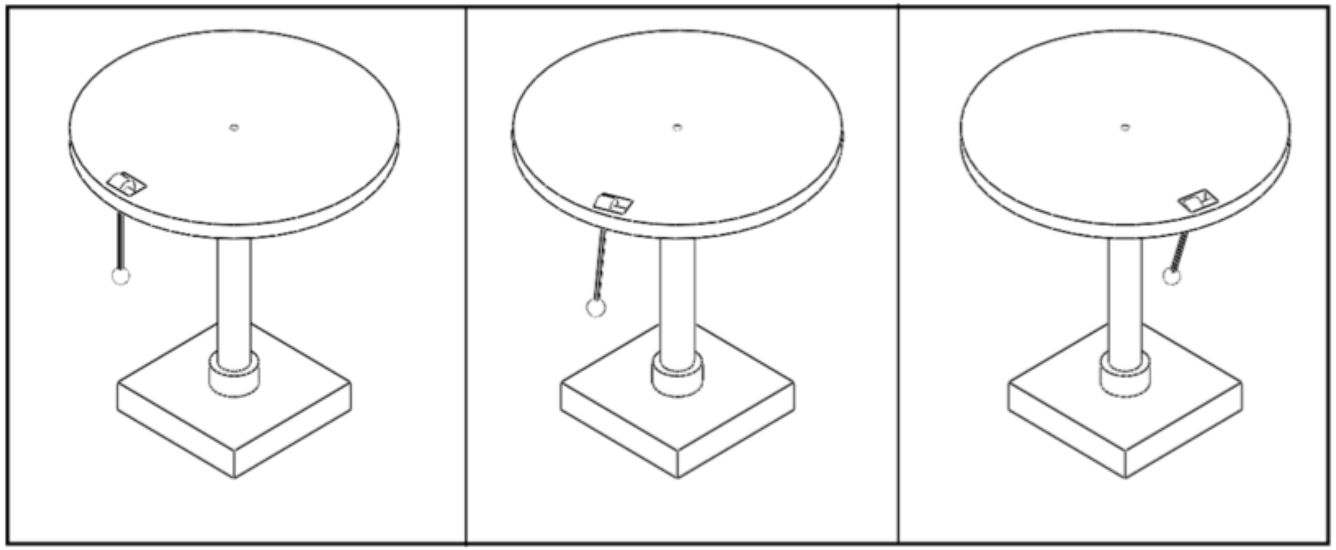
$$mL\ddot{\theta} + mg\, s\theta - mL\Omega^2\,(s\psi c\theta - c\psi s\theta)(c\psi c\theta + s\psi s\theta) = 0$$

$$mL\dot{\theta}^2 + mg\, c\theta + L\Omega^2\,(c\psi c\theta + s\psi s\theta)(c\psi c\theta + s\psi s\theta) = T$$

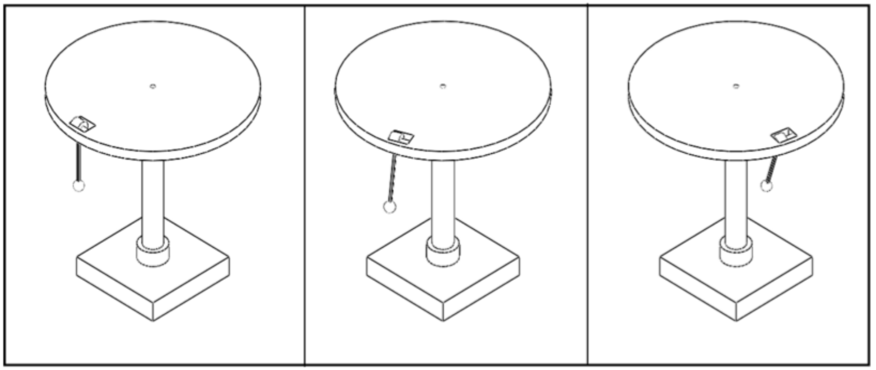
$$-2mL\dot{\theta}\Omega\,(s\psi c\theta - c\psi s\theta) = 0$$

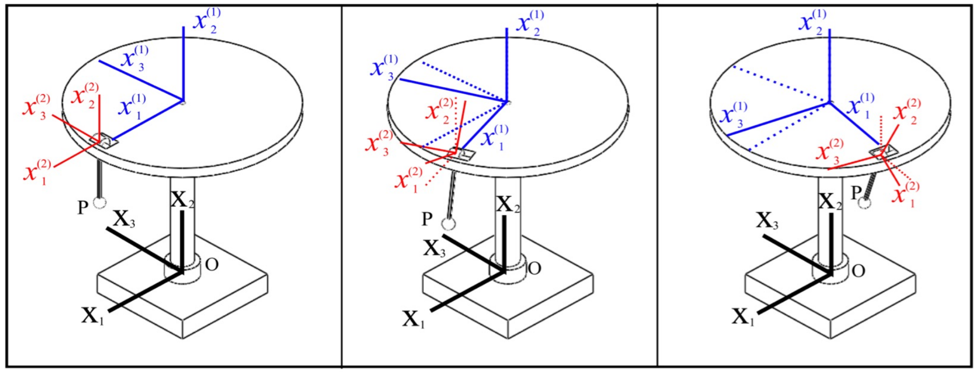


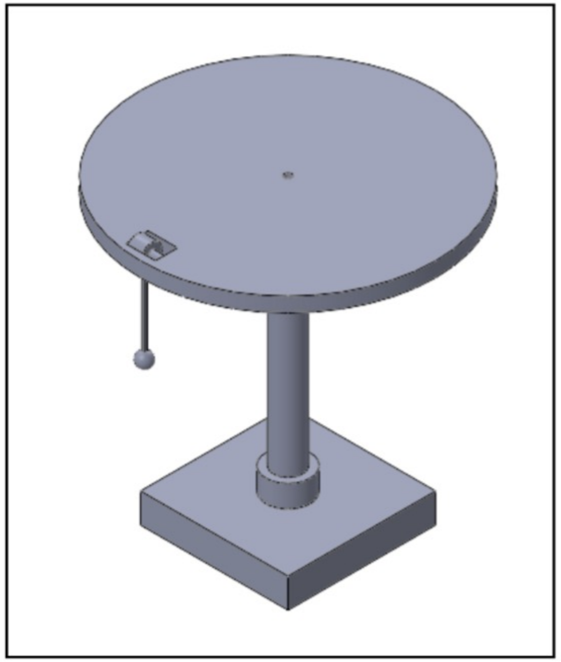
Sistema Disco-Pêndulo - Tipo 2-3

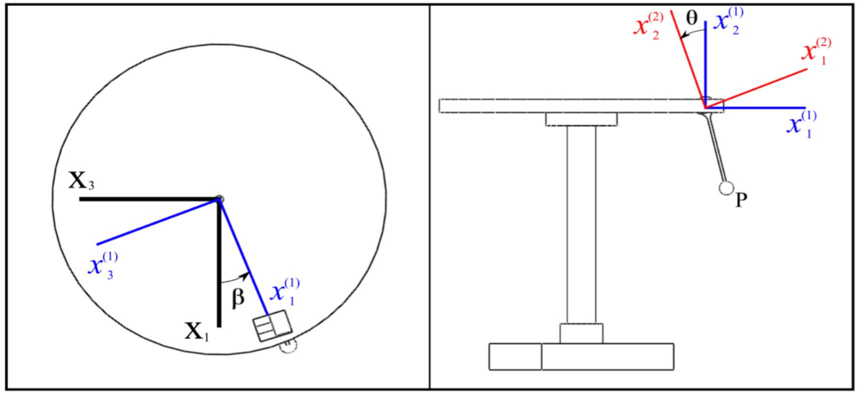


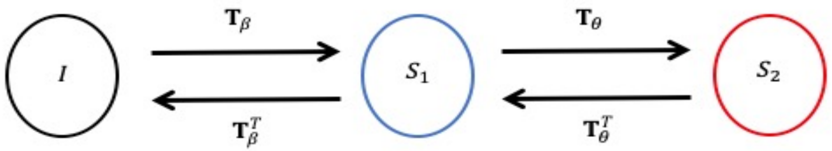
Sistema Disco-Pêndulo - Tipo 2-3









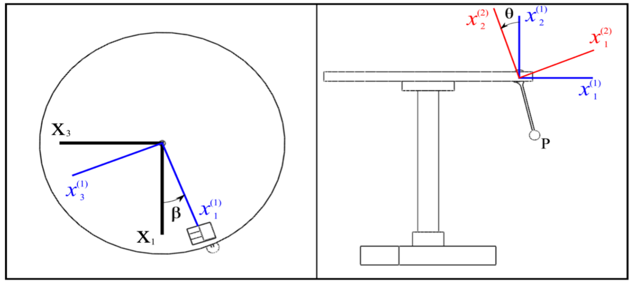


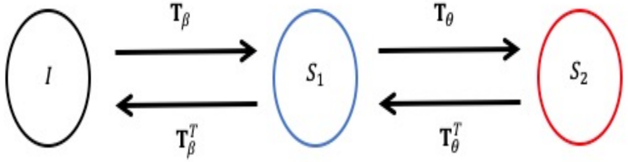
- Inercial, I (X_i)
- Móvel 1, solidário ao disco, S_1 ($x_i^{(1)}$)
- Móvel 2, solidário à haste do pêndulo, S_2 ($x_i^{(2)}$)

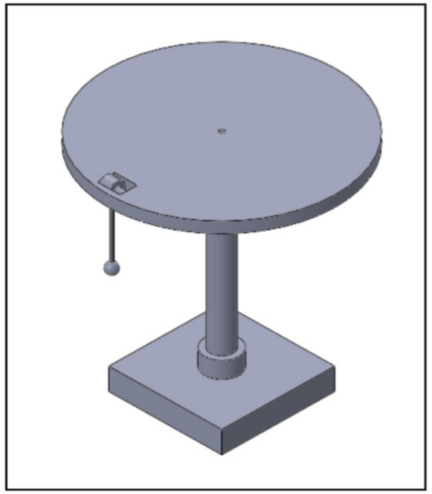
Sistema S1: $$\mathbf{T}_\beta = \begin{bmatrix} c\beta & 0 & -s\beta \\ 0 & 1 & 0 \\ s\beta & 0 & c\beta \end{bmatrix}$$

Sistema S_2: $$\mathbf{T}_\theta = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sistema Disco-Pêndulo - Tipo 2-3: Velocidade e Aceleração Angulares







$$
{}_{S_2}^{I}\boldsymbol{\omega}^{S_2} = \mathbf{T}_\theta \mathbf{T}_\beta\; {}_{I}^{I}\boldsymbol{\omega}^{S_1} + \mathbf{T}_\theta\; {}_{S_1}^{S_1}\boldsymbol{\omega}^{S_2}
$$

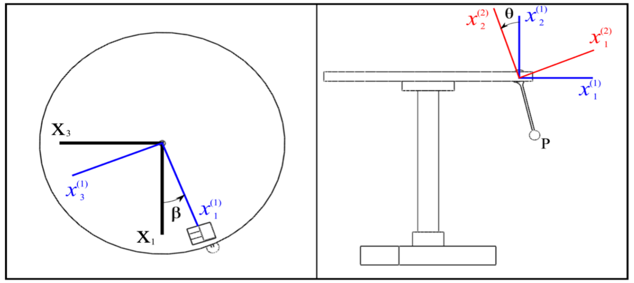
$$
{}_{S_2}^{I}\boldsymbol{\omega}^{S_2} = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & -s\beta \\ 0 & 1 & 0 \\ s\beta & 0 & c\beta \end{bmatrix} \begin{Bmatrix} 0 \\ \dot{\beta} \\ 0 \end{Bmatrix} + \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} = \begin{Bmatrix} \dot{\beta}\, s\theta \\ \dot{\beta}\, c\theta \\ \dot{\theta} \end{Bmatrix}
$$

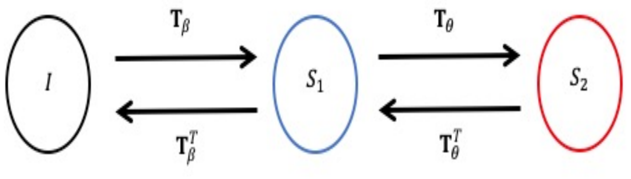
$$
{}_{S_2}^{I}\boldsymbol{\alpha}^{S_2} = \mathbf{T}_\theta \left(\mathbf{T}_\beta\; {}_{I}^{I}\boldsymbol{\alpha}^{S_1}\right) + \mathbf{T}_\theta\; {}_{S_1}^{S_1}\boldsymbol{\alpha}^{S_2} + {}_{S_2}^{I}\boldsymbol{\omega}^{S_1} \times {}_{S_2}^{S_1}\boldsymbol{\omega}^{S_2}
$$

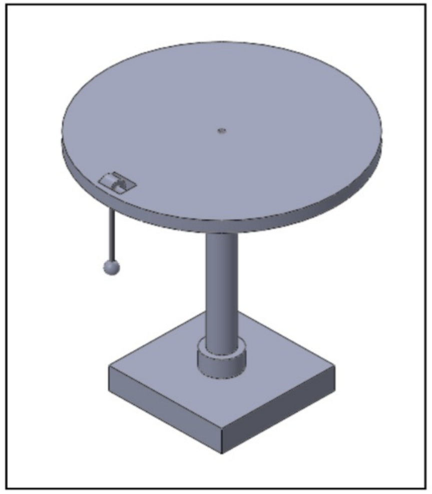
$$
{}_{S_2}^{I}\boldsymbol{\alpha}^{S_2} = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & -s\beta \\ 0 & 1 & 0 \\ s\beta & 0 & c\beta \end{bmatrix} \begin{Bmatrix} 0 \\ \ddot{\beta} \\ 0 \end{Bmatrix} + \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \ddot{\theta} \end{Bmatrix} + \begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ \dot{\beta}\, s\theta & \dot{\beta}\, c\theta & 0 \\ 0 & 0 & \dot{\theta} \end{vmatrix}
$$

$$
{}_{S_2}^{I}\boldsymbol{\alpha}^{S_2} = \begin{Bmatrix} \ddot{\beta}\, s\theta \\ \ddot{\beta}\, c\theta \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \ddot{\theta} \end{Bmatrix} + \begin{Bmatrix} \dot{\beta}\dot{\theta}\, c\theta \\ -\dot{\beta}\dot{\theta}\, c\theta \\ 0 \end{Bmatrix} = \begin{Bmatrix} \ddot{\beta}\, s\theta + \dot{\beta}\dot{\theta}\, c\theta \\ \ddot{\beta}\, c\theta - \dot{\beta}\dot{\theta}\, s\theta \\ \ddot{\theta} \end{Bmatrix}
$$

Sistema Disco-Pêndulo - Tipo 2-3: Aceleração







$$ {}^I_{S_2}\mathbf{a}^P = \boxed{{}^I_{S_2}\mathbf{a}^{O_2}} + \cancelto{0}{{}^{S_2}_{S_2}\mathbf{a}^P} + {}^I_{S_2}\boldsymbol{\alpha}^{S_2}\times{}^{S_2}_{S_2}\mathbf{p}^P + {}^I_{S_2}\boldsymbol{\omega}^{S_2}\times\left({}^I_{S_2}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_2}\mathbf{p}^P\right) + 2\,{}^I_{S_2}\boldsymbol{\omega}^{S_2}\times\cancelto{0}{{}^{S_2}_{S_2}\mathbf{v}^P} $$

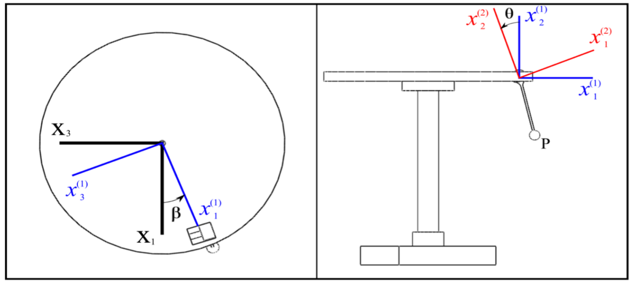
$$ {}^I_{S_2}\mathbf{a}^{O_2} = {}^I_{S_2}\boldsymbol{\alpha}^{S_1}\times{}^{S_1}_{S_2}\mathbf{p}^{O_1} + {}^I_{S_2}\boldsymbol{\omega}^{S_1}\times\left({}^I_{S_2}\boldsymbol{\omega}^{S_1}\times{}^{S_1}_{S_2}\mathbf{p}^{O_1}\right) $$

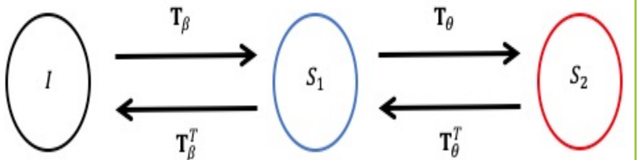
$$ {}^{S_1}_{S_2}\mathbf{p}^{O_1} = \mathbf{T}_\theta\,{}^{S_1}_{S_1}\mathbf{p}^{O_1} = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} R \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} R\,c\theta \\ -R\,s\theta \\ 0 \end{Bmatrix} $$

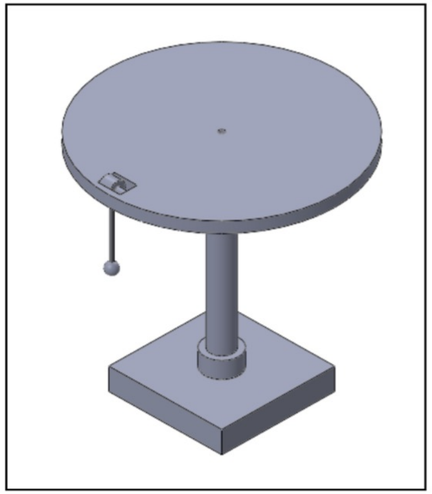
$$ {}^I_{S_2}\boldsymbol{\alpha}^{S_1}\times{}^{S_1}_{S_2}\mathbf{p}^{O_1} = \begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ \ddot{\beta}\,s\theta + \dot{\beta}\dot{\theta}\,c\theta & \ddot{\beta}\,c\theta - \dot{\beta}\dot{\theta}\,s\theta & 0 \\ R\,c\theta & -R\,s\theta & 0 \end{vmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -R\ddot{\beta} \end{Bmatrix} $$

$$ {}^I_{S_2}\boldsymbol{\omega}^{S_1}\times\left({}^I_{S_2}\boldsymbol{\omega}^{S_1}\times{}^{S_1}_{S_2}\mathbf{p}^{O_1}\right) = {}^I_{S_2}\boldsymbol{\omega}^{S_1}\times\begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ \dot{\beta}\,s\theta & \dot{\beta}\,c\theta & 0 \\ R\,c\theta & -R\,s\theta & 0 \end{vmatrix} = \begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ \dot{\beta}\,s\theta & \dot{\beta}\,c\theta & 0 \\ 0 & 0 & -R\dot{\beta} \end{vmatrix} = \begin{Bmatrix} -R\dot{\beta}^2\,c\theta \\ R\dot{\beta}^2\,s\theta \\ 0 \end{Bmatrix} $$

Sistema Disco-Pêndulo - Tipo 2-3: Aceleração







$$ {}^{I}_{S_2}\mathbf{a}^{P} = {}^{I}_{S_2}\mathbf{a}^{O_2} + \cancelto{0}{{}^{S_2}_{S_2}\mathbf{a}^{P}} + {}^{I}_{S_2}\boldsymbol{\alpha}^{S_2}\times{}^{S_2}_{S_2}\mathbf{p}^{P} + {}^{I}_{S_2}\boldsymbol{\omega}^{S_2}\times\left({}^{I}_{S_2}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_2}\mathbf{p}^{P}\right) + 2\,{}^{I}_{S_2}\boldsymbol{\omega}^{S_2}\times\cancelto{0}{{}^{S_2}_{S_2}\mathbf{v}^{P}} $$

$$ {}^{S_2}_{S_2}\mathbf{p}^{P} = \begin{Bmatrix} 0 \\ -L \\ 0 \end{Bmatrix} $$

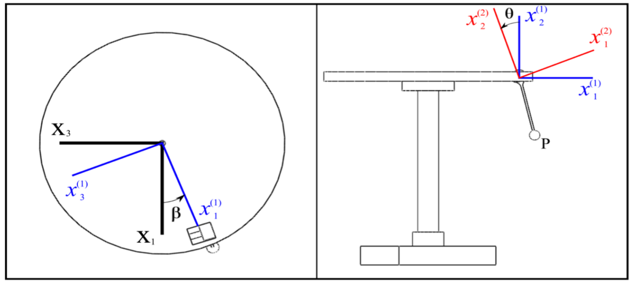
$$ {}^{I}_{S_2}\boldsymbol{\alpha}^{S_2}\times{}^{S_2}_{S_2}\mathbf{p}^{P} = \begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ \ddot{\beta}\ s\theta + \dot{\beta}\dot{\theta}\ c\theta & \ddot{\beta}\ c\theta - \dot{\beta}\dot{\theta}\ s\theta & \ddot{\theta} \\ 0 & -L & 0 \end{vmatrix} = \begin{Bmatrix} L\ddot{\theta} \\ 0 \\ -L\ddot{\beta}\ s\theta - L\dot{\beta}\dot{\theta}\ c\theta \end{Bmatrix} $$

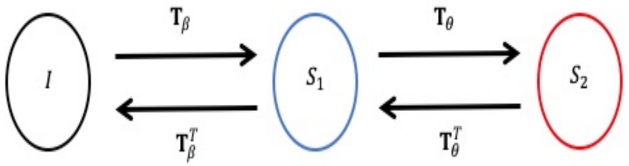
$$ {}^{I}_{S_2}\boldsymbol{\omega}^{S_2}\times\left({}^{I}_{S_2}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_2}\mathbf{p}^{P}\right) = {}^{I}_{S_2}\boldsymbol{\omega}^{S_2}\times\begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ \dot{\beta}\ s\theta & \dot{\beta}\ c\theta & \dot{\theta} \\ 0 & -L & 0 \end{vmatrix} = \begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ \dot{\beta}\ s\theta & \dot{\beta}\ c\theta & \dot{\theta} \\ L\dot{\theta} & 0 & -L\dot{\beta}\ s\theta \end{vmatrix} = \begin{Bmatrix} -L\dot{\beta}^2\ c\theta s\theta \\ L\dot{\beta}^2\ s^2\theta + L\dot{\theta}^2 \\ -L\dot{\beta}\dot{\theta}\ c\theta \end{Bmatrix} $$

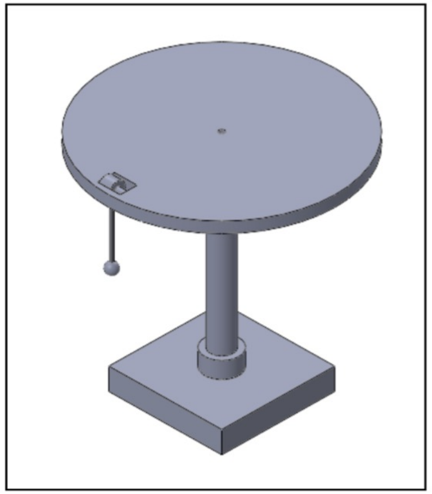
Sistema Disco-Pêndulo - Tipo 2-3: Aceleração

$$
{}^{I}_{S_2}\mathbf{a}^{P} = {}^{I}_{S_2}\mathbf{a}^{O_2} + \underbrace{\cancel{{}^{S_2}_{S_2}\mathbf{a}^{P}}}_{0} + {}^{I}_{S_2}\boldsymbol{\alpha}^{S_2}\times{}^{S_2}_{S_2}\mathbf{p}^{P} + {}^{I}_{S_2}\boldsymbol{\omega}^{S_2}\times\left({}^{I}_{S_2}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_2}\mathbf{p}^{P}\right) + \underbrace{\cancel{2\,{}^{I}_{S_2}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_2}\mathbf{v}^{P}}}_{0}
$$

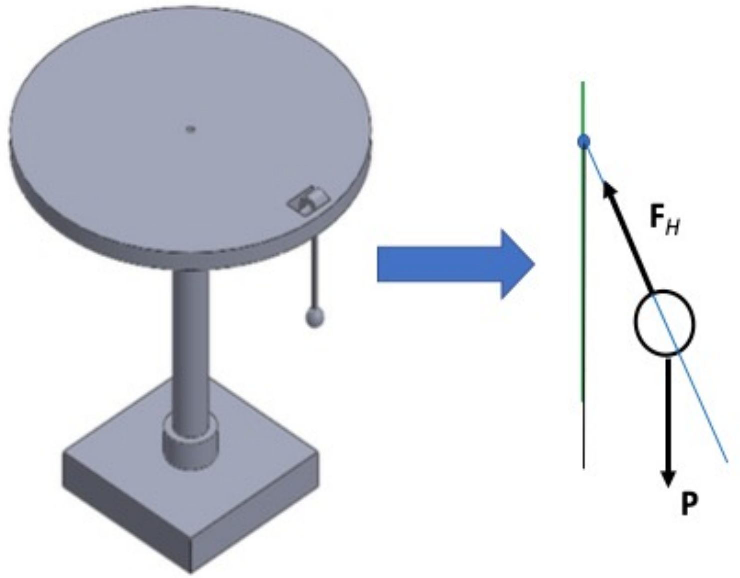
$$
\boxed{{}^{I}_{S_2}\mathbf{a}^{P} = \begin{Bmatrix} R\dot{\beta}^2\, c\theta + L\ddot{\theta} - L\dot{\beta}^2\, c\theta s\theta \\ R\dot{\beta}^2\, s\theta + L\dot{\beta}^2\, s^2\theta + L\dot{\theta}^2 \\ -R\ddot{\beta} - L\ddot{\beta}\, s\theta - 2L\dot{\beta}\dot{\theta}\, c\theta \end{Bmatrix}}
$$

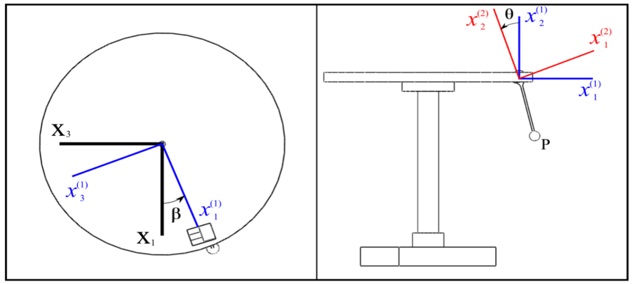


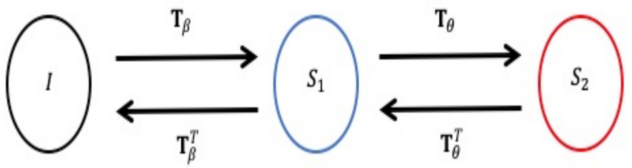


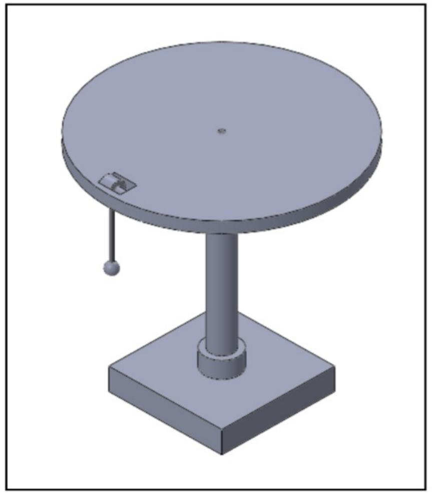


Sistema Disco-Pêndulo - Tipo 2-3: Forças





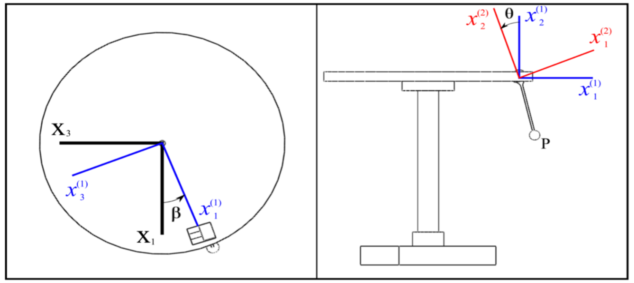


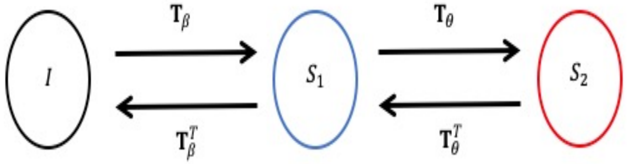


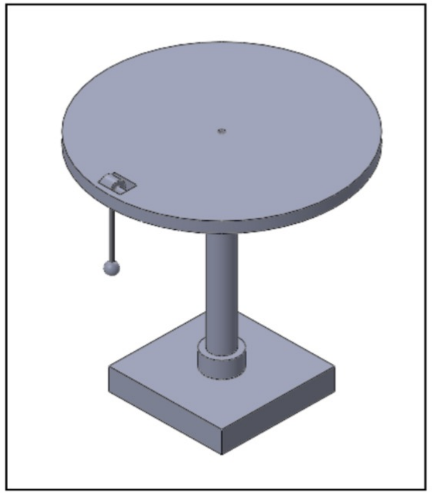
$$
{}_{S_2}\mathbf{P} = \mathbf{T}_\theta \mathbf{T}_\beta \; {}_I\mathbf{P} = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & -s\beta \\ 0 & 1 & 0 \\ s\beta & 0 & c\beta \end{bmatrix} \begin{Bmatrix} 0 \\ -mg \\ 0 \end{Bmatrix} = \begin{Bmatrix} -mg\; s\theta \\ -mg\; c\theta \\ 0 \end{Bmatrix}
$$

$$
{}_{S_2}\mathbf{F}_H = \begin{Bmatrix} 0 \\ N_2 \\ 0 \end{Bmatrix}
$$

Sistema Disco-Pêndulo - Tipo 2-3: Equações de Movimento







$$m\,{}_{s_2}^{I}\mathbf{a}^{P} = {}_{s_2}\mathbf{F}$$

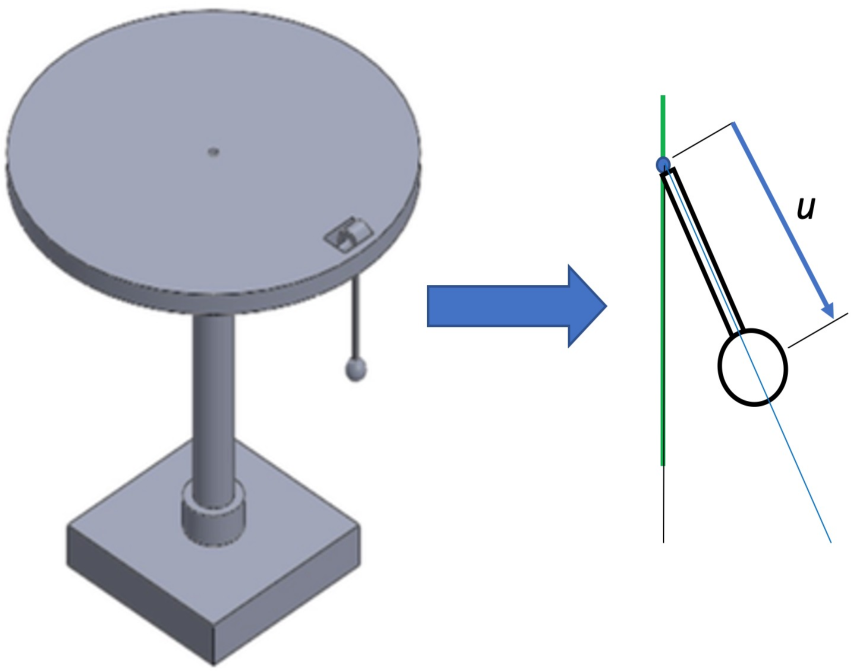
$$m\begin{Bmatrix} R\dot{\beta}^2\,c\theta + L\ddot{\theta} - L\dot{\beta}^2\,c\theta s\theta \\ R\dot{\beta}^2\,s\theta + L\dot{\beta}^2\,s^2\theta + L\dot{\theta}^2 \\ -R\ddot{\beta} - L\ddot{\beta}\,s\theta - 2L\dot{\beta}\dot{\theta}\,c\theta \end{Bmatrix} = \begin{Bmatrix} -mg\,s\theta \\ N_2 - mg\,c\theta \\ 0 \end{Bmatrix}$$

$$L\ddot{\theta} + g\,s\theta + R\dot{\beta}^2\,c\theta - L\dot{\beta}^2\,c\theta s\theta = 0$$

$$R\dot{\beta}^2\,s\theta + L\dot{\beta}^2\,s^2\theta - L\dot{\theta}^2 + g\,c\theta = \frac{N_2}{m}$$

$$R\ddot{\beta} + L\ddot{\beta}\,s\theta + 2L\dot{\beta}\dot{\theta}\,c\theta = 0$$

Sistema Disco-Pêndulo com Haste Flexível - Tipo 2-3: Aceleração

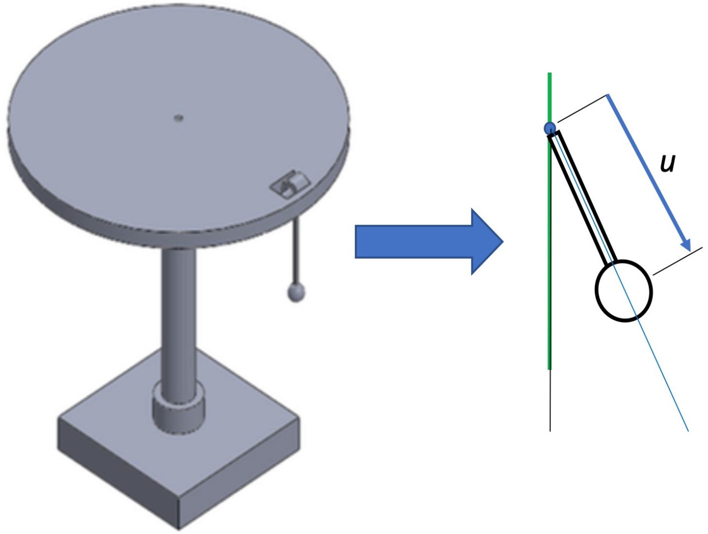


$$I \underset{\mathbf{T}_\beta^T}{\overset{\mathbf{T}_\beta}{\rightleftarrows}} S_1 \underset{\mathbf{T}_\theta^T}{\overset{\mathbf{T}_\theta}{\rightleftarrows}} S_2$$

Inclusão de um GdL adicional!

$$ {}^{I}_{S_2}\mathbf{a}^{P} = {}^{I}_{S_2}\mathbf{a}^{O_2} + \boxed{{}^{S_2}_{S_2}\mathbf{a}^{P}} + {}^{I}_{S_2}\boldsymbol{\alpha}^{S_2}\times{}^{S_2}_{S_2}\mathbf{p}^{P} + {}^{I}_{S_2}\boldsymbol{\omega}^{S_2}\times\left({}^{I}_{S_2}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_2}\mathbf{p}^{P}\right) + \boxed{2\,{}^{I}_{S_2}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_2}\mathbf{v}^{P}} $$

Sistema Disco-Pêndulo com Haste Flexível - Tipo 2-3: Aceleração



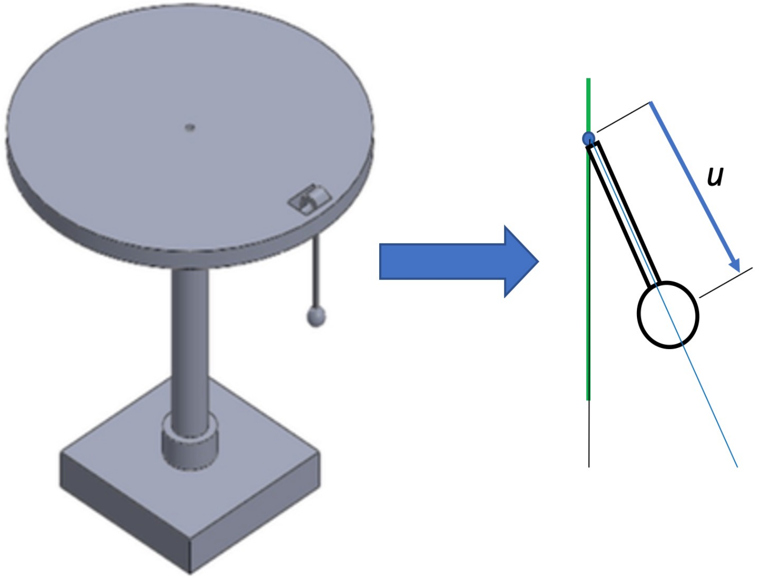
$$ {}^{I}_{S_2}\mathbf{a}^{P} = {}^{I}_{S_2}\mathbf{a}^{O_2} + \boxed{{}^{S_2}_{S_2}\mathbf{a}^{P}} + {}^{I}_{S_2}\boldsymbol{\alpha}^{S_2} \times {}^{S_2}_{S_2}\mathbf{p}^{P} + {}^{I}_{S_2}\boldsymbol{\omega}^{S_2} \times \left({}^{I}_{S_2}\boldsymbol{\omega}^{S_2} \times {}^{S_2}_{S_2}\mathbf{p}^{P} \right) + \boxed{2\,{}^{I}_{S_2}\boldsymbol{\omega}^{S_2} \times {}^{S_2}_{S_2}\mathbf{v}^{P}} $$

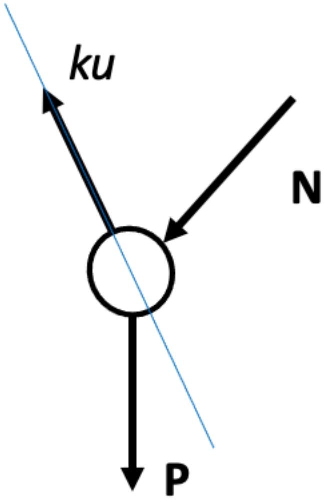
$$ {}^{S_2}_{S_2}\mathbf{a}^{P} = \begin{Bmatrix} 0 \\ -\ddot{u} \\ 0 \end{Bmatrix} $$

$$ 2\,{}^{I}_{S_2}\boldsymbol{\omega}^{S_2} \times {}^{S_2}_{S_2}\mathbf{v}^{P} = 2 \begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ \dot{\beta}\, s\theta & \dot{\beta}\, c\theta & \dot{\theta} \\ 0 & -\dot{u} & 0 \end{vmatrix} = \begin{Bmatrix} 2\dot{u}\dot{\theta} \\ 0 \\ -2\dot{u}\dot{\beta}\, s\theta \end{Bmatrix} $$

$$ \boxed{ {}^{I}_{S_2}\mathbf{a}^{P} = \begin{Bmatrix} R\dot{\beta}^2\, c\theta + u\ddot{\theta} - u\dot{\beta}^2\, c\theta s\theta + 2\dot{u}\dot{\theta} \\ R\dot{\beta}^2\, s\theta + u\dot{\beta}^2\, s^2\theta + u\dot{\theta}^2 - \ddot{u} \\ -R\ddot{\beta} - u\ddot{\beta}\, s\theta - 2u\dot{\beta}\dot{\theta}\, c\theta - 2\dot{u}\dot{\beta}\, s\theta \end{Bmatrix} } $$

Sistema Disco-Pêndulo com Haste Flexível - Tipo 2-3: Forças

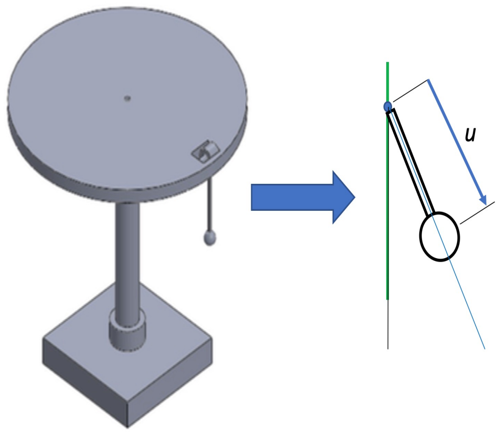




$$ {}_{S_2}\mathbf{F}_E = \begin{Bmatrix} N_1 \\ ku \\ N_3 \end{Bmatrix} $$

$$ {}_{S_2}\mathbf{P} = \mathbf{T}_\theta \mathbf{T}_\beta \ {}_I\mathbf{P} = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & -s\beta \\ 0 & 1 & 0 \\ s\beta & 0 & c\beta \end{bmatrix} \begin{Bmatrix} 0 \\ -mg \\ 0 \end{Bmatrix} = \begin{Bmatrix} -mg\ s\theta \\ -mg\ c\theta \\ 0 \end{Bmatrix} $$

Sistema Disco-Pêndulo com Haste Flexível – Tipo 2-3: Equações de Movimento



$$m\begin{Bmatrix} R\dot{\beta}^2\, c\theta + u\ddot{\theta} - u\dot{\beta}^2\, c\theta s\theta + 2\dot{u}\dot{\theta} \\ R\dot{\beta}^2\, s\theta + \, u\dot{\beta}^2\, s^2\theta + u\dot{\theta}^2 - \ddot{u} \\ -R\ddot{\beta} - u\ddot{\beta}\, s\theta - 2u\dot{\beta}\dot{\theta}\, c\theta - 2\dot{u}\dot{\beta}\, s\theta \end{Bmatrix} = \begin{Bmatrix} N_1 - mg\, s\theta \\ ku - mg\, c\theta \\ N_3 \end{Bmatrix}$$

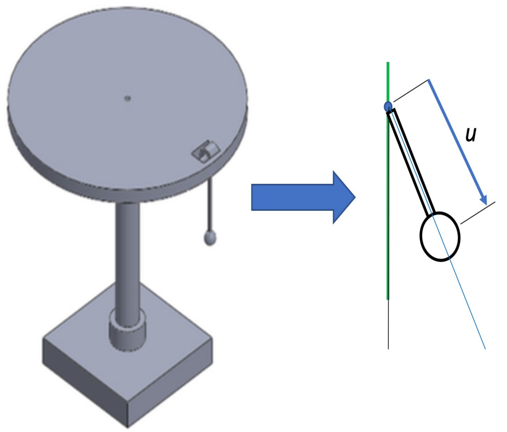
Pêndulo é travado em uma posição $\theta = \theta_0$ e portanto $\ddot{\theta} = \dot{\theta} = 0$.

Admite-se que $\ddot{\beta} = 0$

$$m\begin{Bmatrix} R\dot{\beta}^2\, c\theta_0 - u\dot{\beta}^2\, c\theta_0 s\theta_0 \\ R\dot{\beta}^2\, s\theta_0 + \, u\dot{\beta}^2\, s^2\theta_0 - \ddot{u} \\ -2\dot{u}\dot{\beta}\, s\theta_0 \end{Bmatrix} = \begin{Bmatrix} N_1 - mg\, s\theta_0 \\ ku - mg\, c\theta_0 \\ N_3 \end{Bmatrix}$$

$$\boxed{m\ddot{u} + \left(k - m\dot{\beta}^2\, s^2\theta_0\right)u = m\left(R\dot{\beta}^2\, s\theta_0 + g\, c\theta_0\right)}$$

Sistema Disco-Pêndulo com Haste Flexível – Tipo 2-3: Equações de Movimento



$$\boxed{m\ddot{u} + \left(k - m\dot{\beta}^2\, s^2\theta_0\right)u = m\left(R\dot{\beta}^2\, s\theta_0 + g\, c\theta_0\right)}$$

$$\ddot{u} + \zeta\dot{u} + \varphi^2\, u = f(t)$$

$$\varphi^2 = \frac{\left(k - m\dot{\beta}^2\, s^2\theta_0\right)}{m}$$

$$\zeta = 0$$

$$f(t) = R\dot{\beta}^2\, s\theta_0 + g\, c\theta_0$$

Solução homogênea: $u_h = Ae^{\lambda t}$

$$\lambda_{1,2} = \pm\mu = \pm\sqrt{\frac{\left(k - m\dot{\beta}^2\, s^2\theta_0\right)}{m}}$$

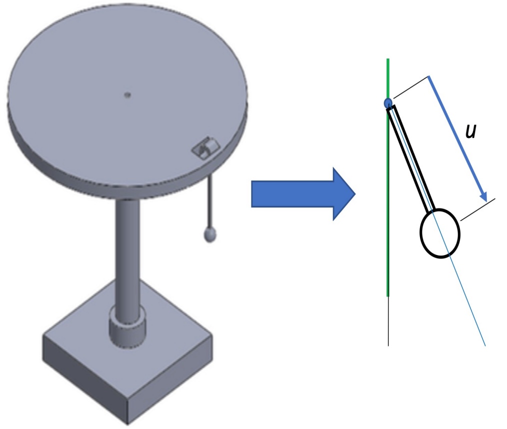
Solução particular: $u_p = K$

$$K = \frac{R\dot{\beta}^2\, s\theta_0 + g\, c\theta_0}{\mu^2} = \frac{m\left(R\dot{\beta}^2\, s\theta_0 + g\, c\theta_0\right)}{k - m\dot{\beta}^2\, s^2\theta_0}$$

Solução geral:

$$\boxed{u = A_1\, e^{\mu t} + A_2\, e^{-\mu t} + \frac{m\left(R\dot{\beta}^2\, s\theta_0 + g\, c\theta_0\right)}{k - m\dot{\beta}^2\, s^2\theta_0}}$$

Sistema Disco-Pêndulo com Haste Flexível - Tipo 2-3: Equações de Movimento



Solução geral:

$$\boxed{u = A_1\, e^{\mu t} + A_2\, e^{-\mu t} + \frac{m\left(R\dot{\beta}^2\, s\theta_0 + g\, c\theta_0\right)}{k - m\dot{\beta}^2\, s^2\theta_0}}$$

Condições iniciais:

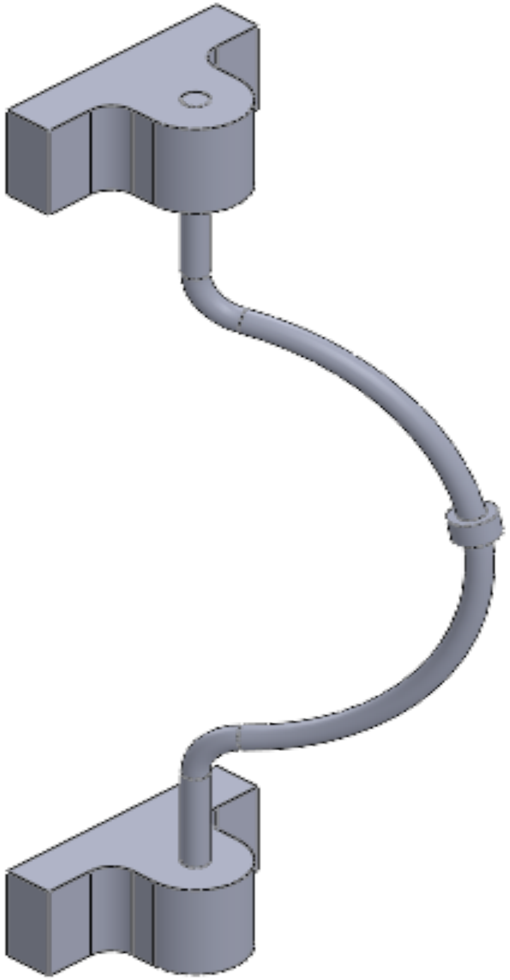
$$u(0) = A_1 + A_2 + \frac{m\left(R\dot{\beta}^2\, s\theta_0 + g\, c\theta_0\right)}{k - m\dot{\beta}^2\, s^2\theta_0} = 0 \quad \Rightarrow \quad A_1 = A_2 = -\frac{m\left(R\dot{\beta}^2\, s\theta_0 + g\, c\theta_0\right)}{2\left(k - m\dot{\beta}^2\, s^2\theta_0\right)}$$

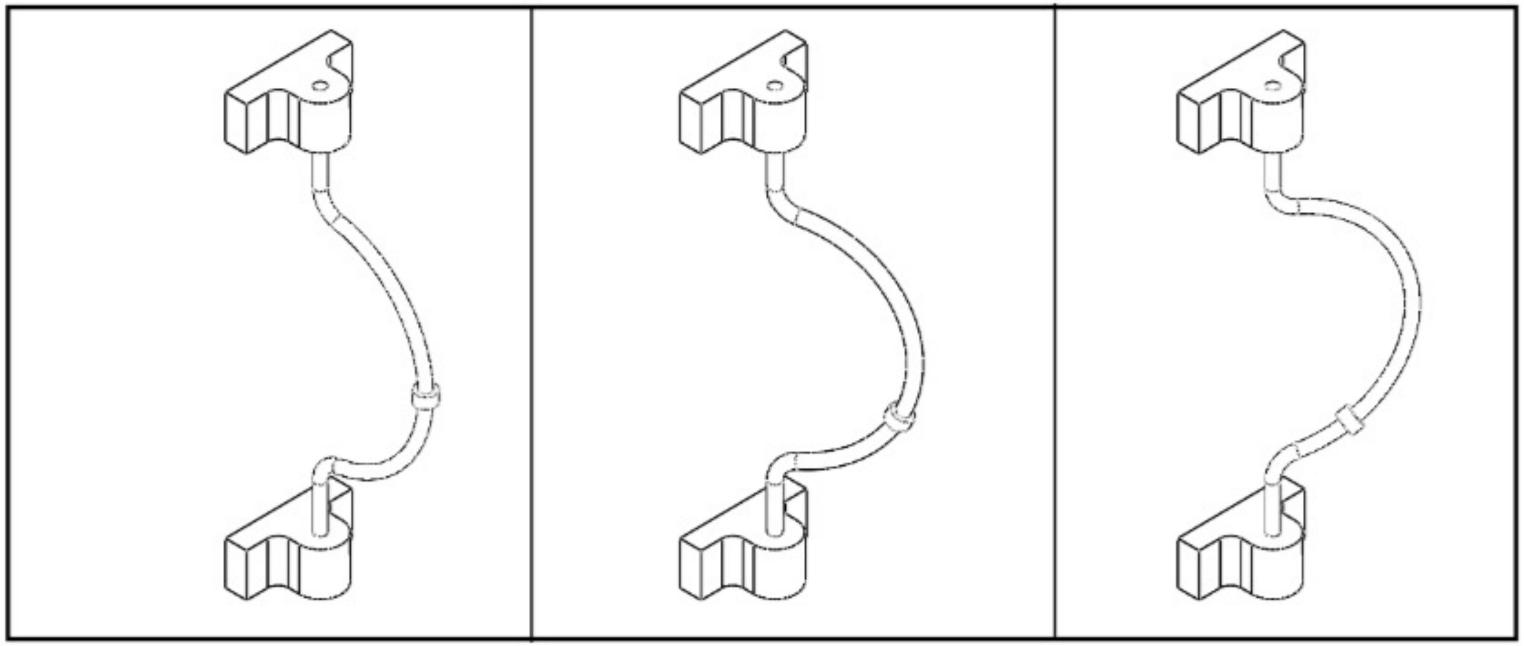
$$\dot{u}(0) = A_1\mu - A_2\mu = (A_1 - A_2)\mu = 0$$

Portanto:

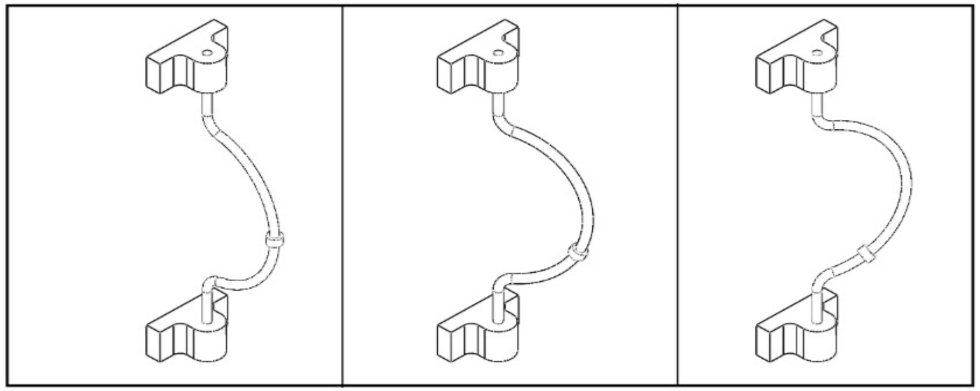
$$\boxed{u = -\frac{m\left(R\dot{\beta}^2\, s\theta_0 + g\, c\theta_0\right)}{k - m\dot{\beta}^2\, s^2\theta_0}\left[\frac{e^{\mu t} + e^{-\mu t}}{2} - 1\right] = -\frac{m(R\dot{\beta}^2\, s\theta_0 + g\, c\theta_0)}{k - m\dot{\beta}^2\, s^2\theta_0}\left[\cosh\left(\sqrt{\frac{(k - m\dot{\beta}^2\, s^2\theta_0)}{m}}\, t\right) - 1\right]}$$

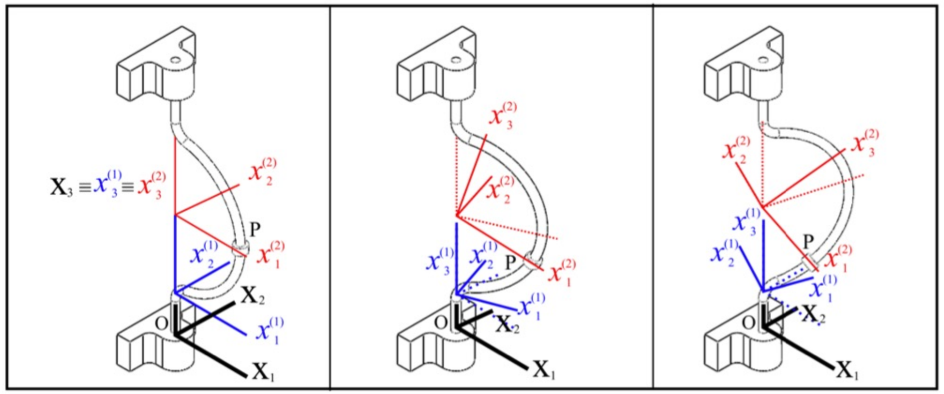
Sistema Tipo 3-2

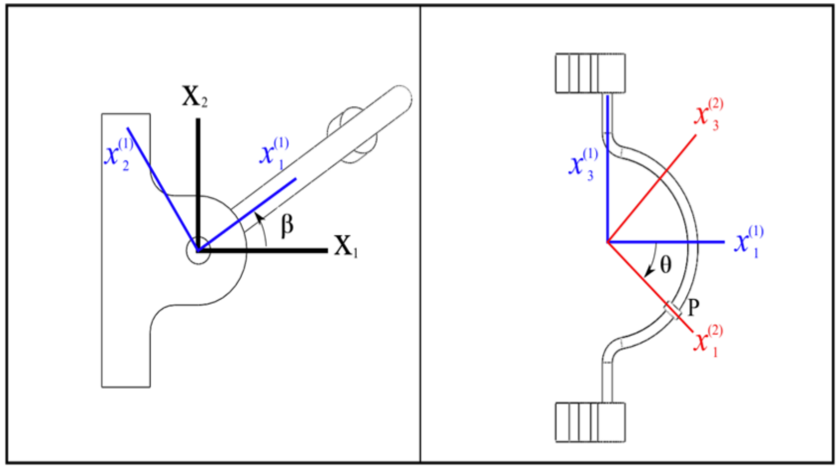




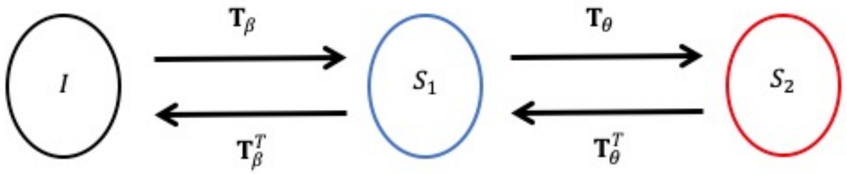
Sistema Tipo 3-2







- Inercial, I (X_i)
- Móvel 1, solidário à haste circular, S_1 ($x_i^{(1)}$)
- Móvel 2, solidário ao corpo deslizante, S_2 ($x_i^{(2)}$)



Sistema S1: $$\mathbf{T}_\beta = \begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sistema S_2: $$\mathbf{T}_\theta = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}$$

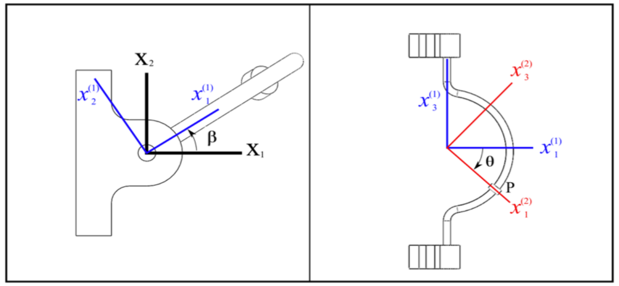
Sistema Tipo 3-2: Velocidade e Aceleração Angulares

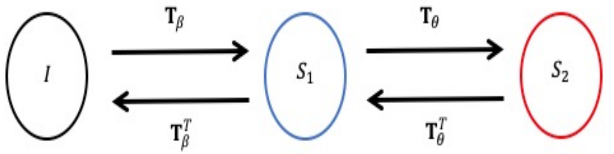
$$ {}^{I}_{S_1}\boldsymbol{\omega}^{S_2} = \mathbf{T}_\beta \, {}^{I}_{I}\boldsymbol{\omega}^{S_1} + {}^{S_1}_{S_1}\boldsymbol{\omega}^{S_2} $$

$$ {}^{I}_{S_1}\boldsymbol{\omega}^{S_2} = \begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\beta} \end{Bmatrix} + \begin{Bmatrix} 0 \\ \dot{\theta} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \dot{\theta} \\ \dot{\beta} \end{Bmatrix} $$

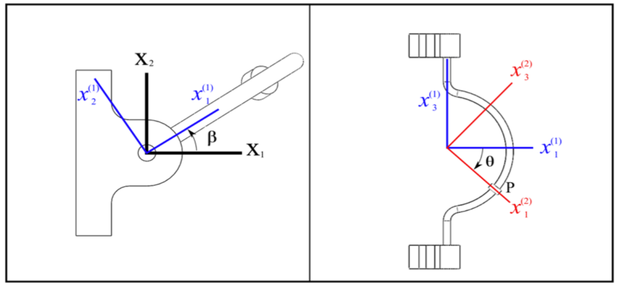
$$ {}^{I}_{S_1}\boldsymbol{\alpha}^{S_2} = \mathbf{T}_\beta \, {}^{I}_{I}\boldsymbol{\alpha}^{S_1} + {}^{S_1}_{S_1}\boldsymbol{\alpha}^{S_2} + {}^{I}_{S_1}\boldsymbol{\omega}^{S_1} \times {}^{S_1}_{S_1}\boldsymbol{\omega}^{S_2} $$

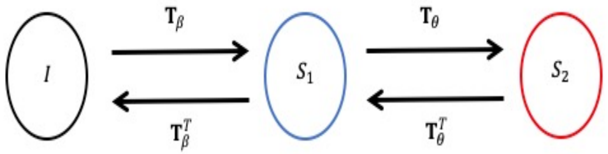
$$ {}^{I}_{S_1}\boldsymbol{\alpha}^{S_2} = \begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \ddot{\beta} \end{Bmatrix} + \begin{Bmatrix} 0 \\ \ddot{\theta} \\ 0 \end{Bmatrix} + \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \dot{\beta} \\ 0 & \dot{\theta} & 0 \end{vmatrix} = \begin{Bmatrix} -\dot{\beta}\dot{\theta} \\ \ddot{\theta} \\ \ddot{\beta} \end{Bmatrix} $$





Sistema Tipo 3-2: Aceleração



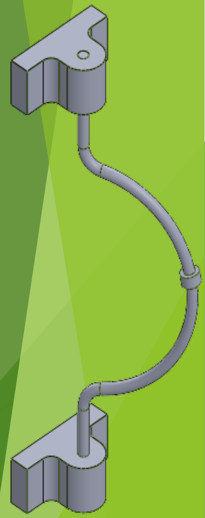


$$ {}^{I}_{S_1}\mathbf{a}^{P} = \cancelto{0}{{}^{I}_{S_1}\mathbf{a}^{O_2}} + \cancelto{0}{{}^{S_2}_{S_1}\mathbf{a}^{P}} + {}^{I}_{S_1}\boldsymbol{\alpha}^{S_2}\times{}^{S_2}_{S_1}\mathbf{p}^{P} + {}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times\left({}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_1}\mathbf{p}^{P}\right) + \cancelto{0}{2\,{}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_1}\mathbf{v}^{P}} $$

$$ {}^{S_2}_{S_1}\mathbf{p}^{P} = \mathbf{T}_\theta^T\;{}^{S_2}_{S_2}\mathbf{p}^{P} = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}\begin{Bmatrix} R \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} R\,c\theta \\ 0 \\ -R\,s\theta \end{Bmatrix} $$

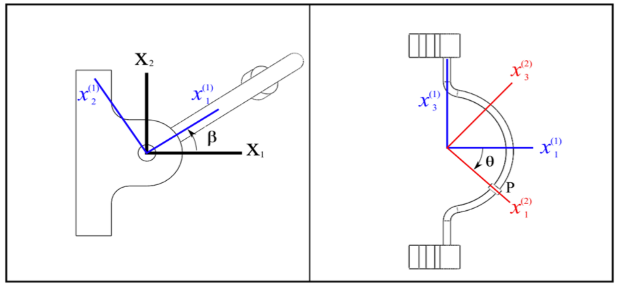
$$ {}^{I}_{S_1}\boldsymbol{\alpha}^{S_2}\times{}^{S_2}_{S_1}\mathbf{p}^{P} = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ -\dot\beta\dot\theta & \ddot\theta & \ddot\beta \\ R\,c\theta & 0 & -R\,s\theta \end{vmatrix} = \begin{Bmatrix} -R\ddot\theta\,s\theta \\ R\ddot\beta\,c\theta - R\dot\beta\dot\theta\,s\theta \\ -R\ddot\theta\,c\theta \end{Bmatrix} $$

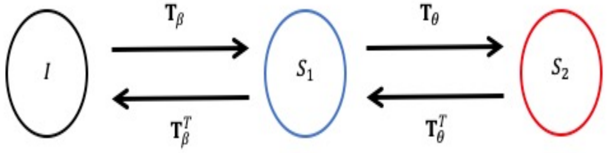
$$ {}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times\left({}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_1}\mathbf{p}^{P}\right) = {}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times\begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ 0 & \dot\theta & \dot\beta \\ R\,c\theta & 0 & -R\,s\theta \end{vmatrix} = \begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ 0 & \dot\theta & \dot\beta \\ -R\dot\theta\,s\theta & R\dot\beta\,c\theta & -R\dot\theta\,c\theta \end{vmatrix} = \begin{Bmatrix} -R\dot\theta^2\,c\theta - R\dot\beta^2\,c\theta \\ -R\dot\theta\dot\beta\,s\theta \\ R\dot\theta^2\,s\theta \end{Bmatrix} $$



$$ \boxed{{}^{I}_{S_1}\mathbf{a}^{P} = \begin{Bmatrix} -R\ddot\theta\,s\theta - R\dot\theta^2\,c\theta - R\dot\beta^2\,c\theta \\ R\ddot\beta\,c\theta - 2R\dot\beta\dot\theta\,s\theta \\ -R\ddot\theta\,c\theta + R\dot\theta^2\,s\theta \end{Bmatrix}} $$

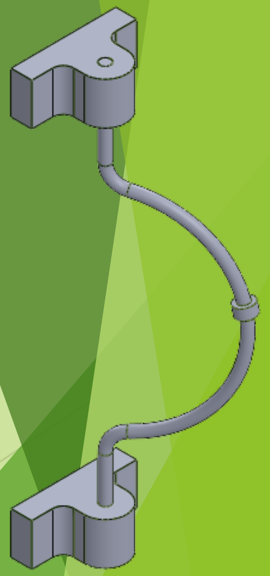
Sistema Tipo 3-2: Forças



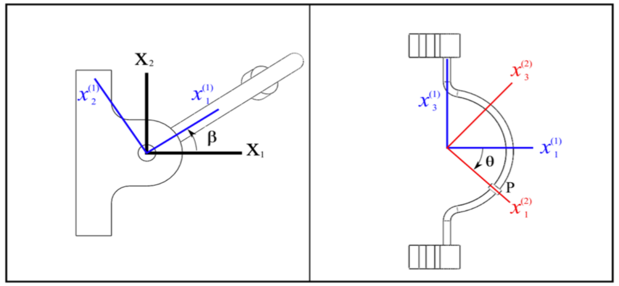


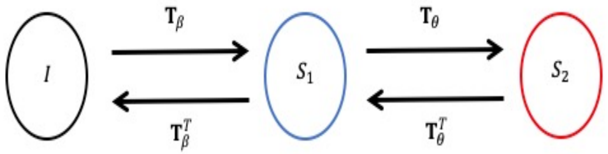
$$
{}_{S_1}\mathbf{P} = \mathbf{T}_\beta \; {}_{I}\mathbf{P} = \begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -mg \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -mg \end{Bmatrix}
$$

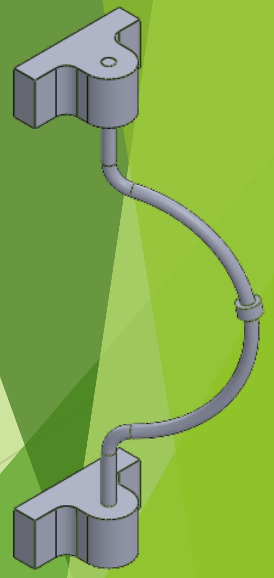
$$
{}_{S_1}\mathbf{N} = \mathbf{T}_\theta^T \; {}_{S_2}\mathbf{N} = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} N_1 \, c\theta \\ N_2 \\ -N_1 \, s\theta \end{Bmatrix}
$$



Sistema Tipo 3-2: Equações de Movimento







$$m\begin{Bmatrix} -R\ddot{\theta}\, s\theta - R\dot{\theta}^2\, c\theta - R\dot{\beta}^2\, c\theta \\ R\ddot{\beta}\, c\theta - 2R\dot{\beta}\dot{\theta}\, s\theta \\ -R\ddot{\theta}\, c\theta + R\dot{\theta}^2\, s\theta \end{Bmatrix} = \begin{Bmatrix} N_1\, c\theta \\ N_2 \\ -N_1\, s\theta - mg \end{Bmatrix}$$

$$R\ddot{\theta}\, s\theta + R\dot{\theta}^2\, c\theta + R\dot{\beta}^2\, c\theta = -\frac{N_1 c\theta}{m}$$

$$R\ddot{\beta}\, c\theta - 2R\dot{\beta}\dot{\theta}\, s\theta = \frac{N_2}{m}$$

$$-R\ddot{\theta}\, c\theta + R\dot{\theta}^2\, s\theta = \frac{-N_1\, s\theta - mg}{m}$$