DINÂMICA: Cinemática

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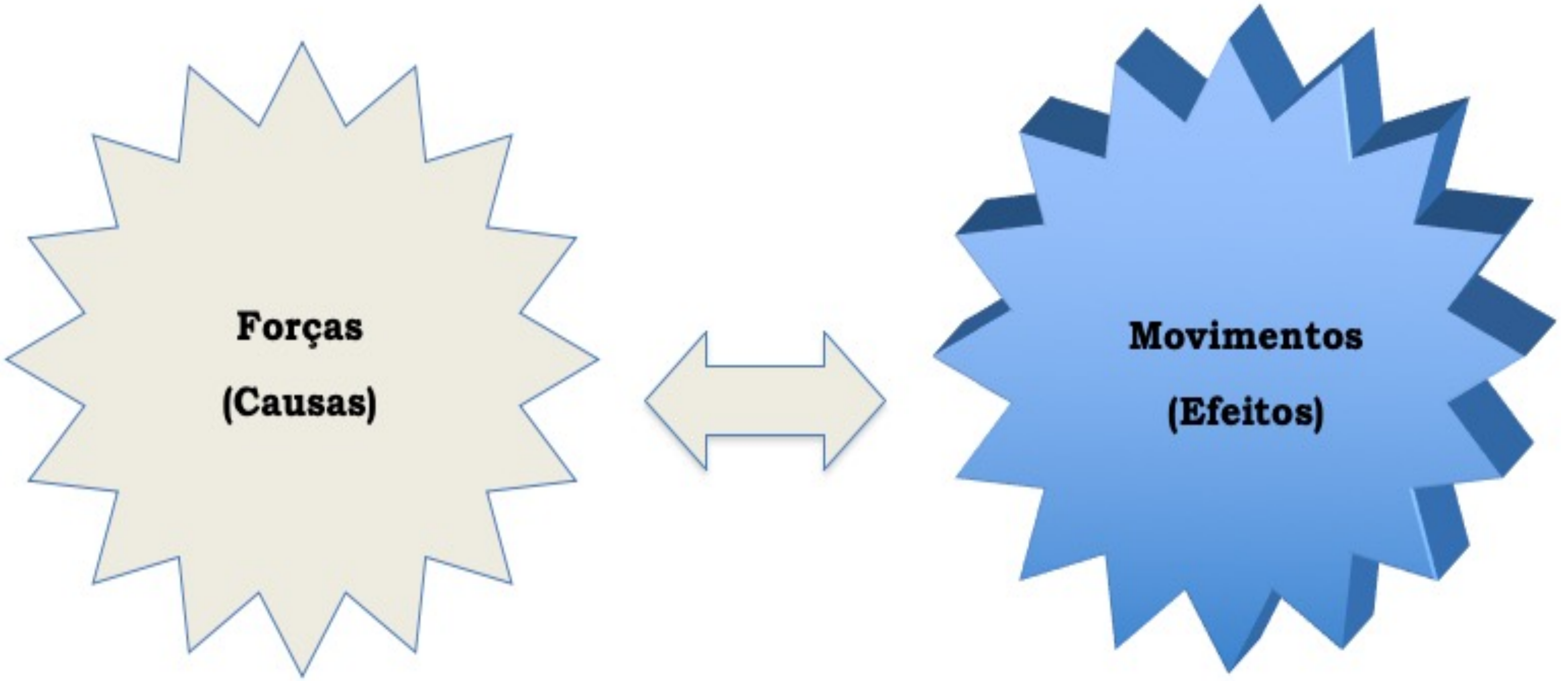
Cinemática

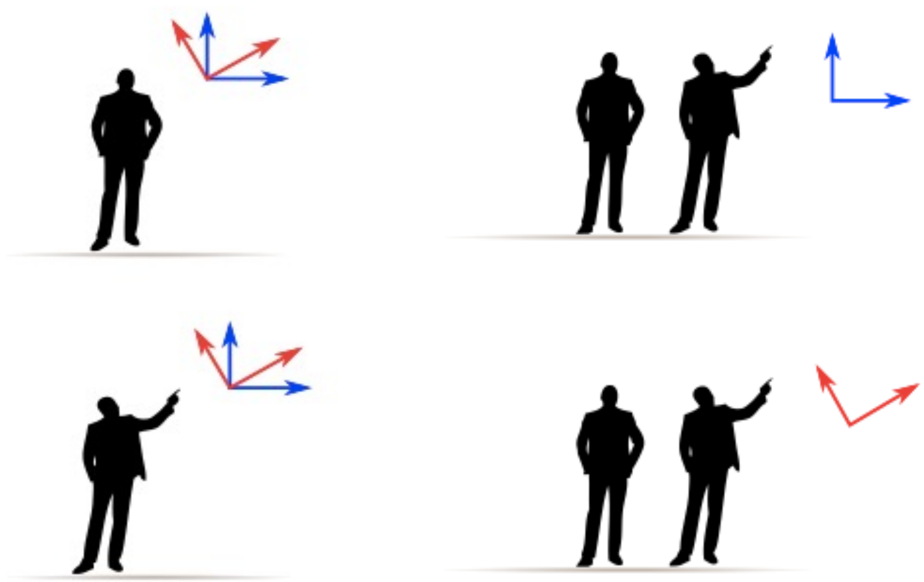
- A cinemática descreve a geometria do movimento não se importando com as suas causas.
- O estado de uma partícula é definido a partir de sua posição e velocidade: grandezas vetoriais.

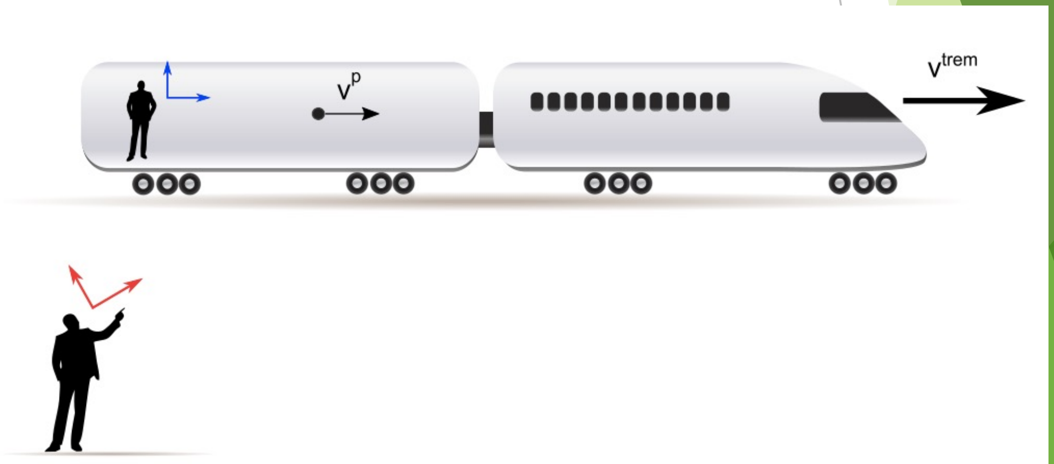


Cinemática

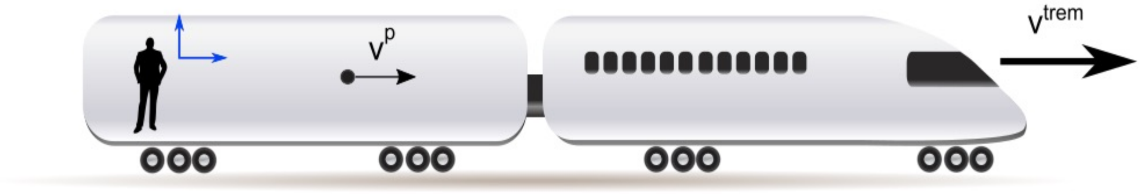
Referencial e Obervador

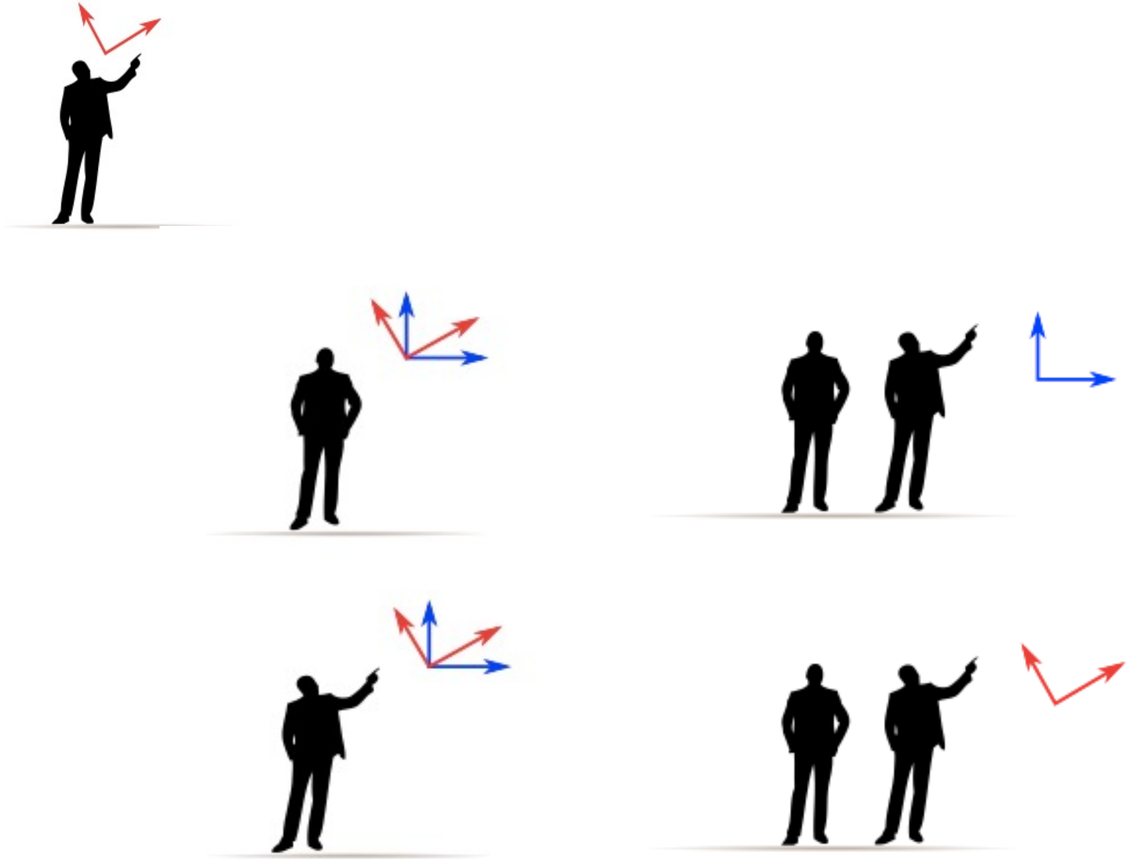
- Referencial: físico
- Sistema coordenado (base): geométrico

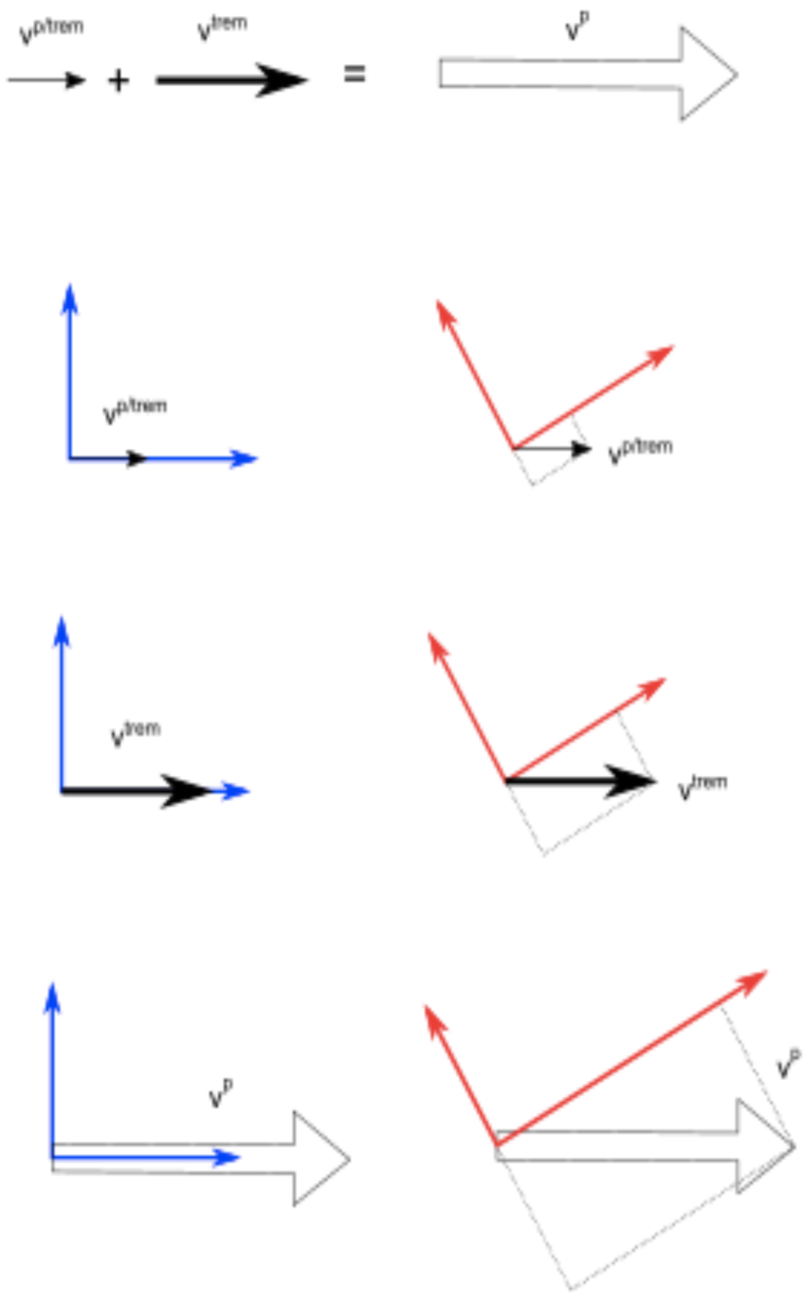




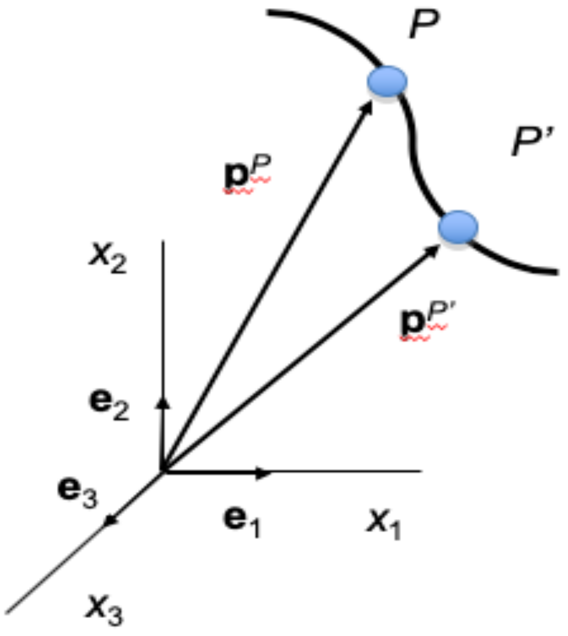
Cinemática







Movimento



Posição: $\mathbf{p}^P = \mathbf{p}^P(t)$

Velocidade: $\mathbf{v}^P = \lim_{\Delta t \to 0} \dfrac{\mathbf{p}^{P'} - \mathbf{p}^P}{\Delta t} = \dfrac{d\mathbf{p}^P}{dt} = \dot{\mathbf{p}}^P$

Aceleração: $\mathbf{a}^P = \lim_{\Delta t \to 0} \dfrac{\mathbf{v}^{P'} - \mathbf{v}^P}{\Delta t} = \dfrac{d\mathbf{v}^P}{dt} = \dot{\mathbf{v}}^P = \dfrac{d^2\mathbf{p}^P}{dt^2} = \ddot{\mathbf{p}}^P$

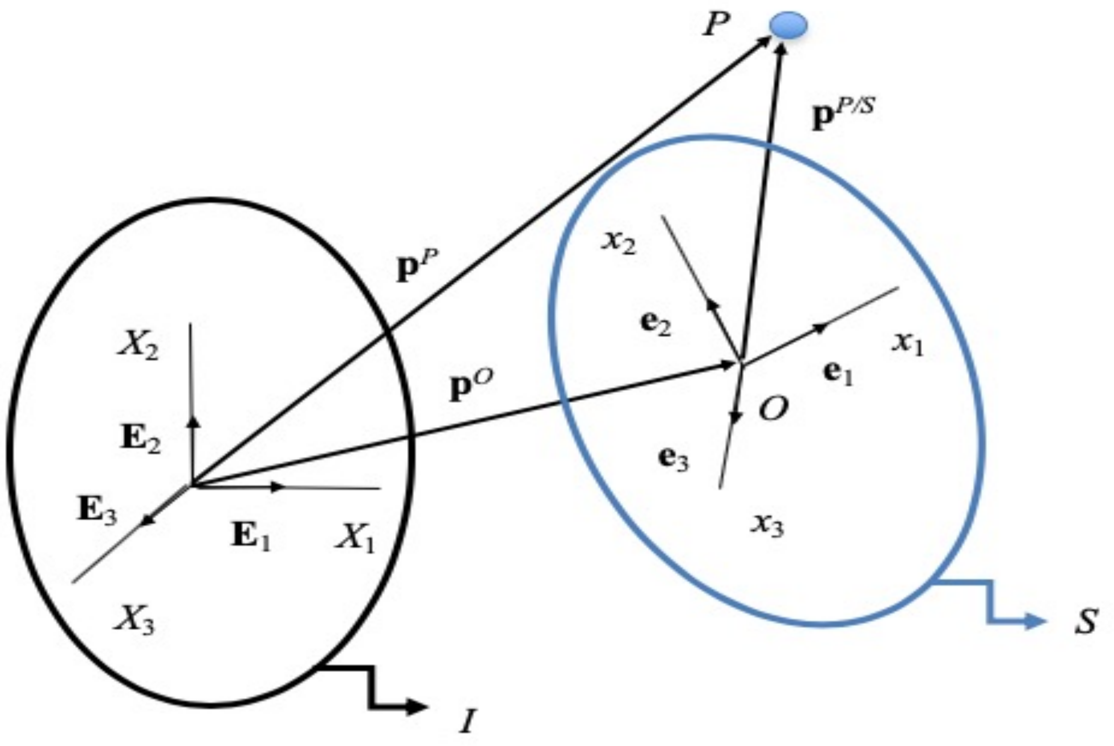
Quantidade de movimento: (*Momentum*) $\mathbf{G}^P = m\mathbf{v}^P$

Quantidade de movimento angular: (*Momentum* angular) $\mathbf{H}^{P/0} = \mathbf{p}^P \times \mathbf{G}^P$

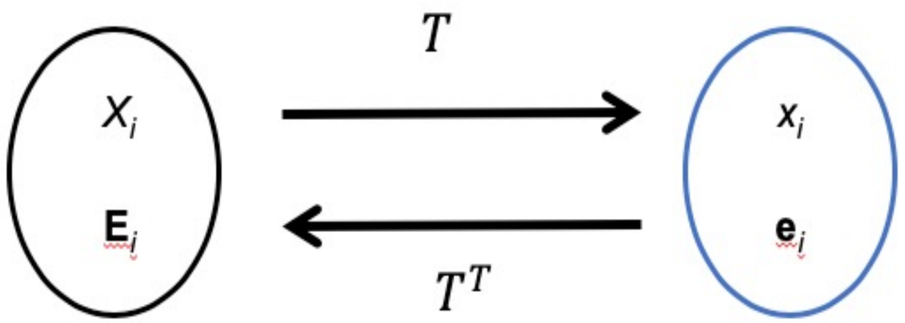
Referenciais Móveis

Descrição do movimento pode ser uma tarefa difícil o que sugere uma abordagem sistemática:

- Decompor o movimento geral em movimentos mais simples.
- Utilizar referenciais móveis, cada um associado a um corpo.

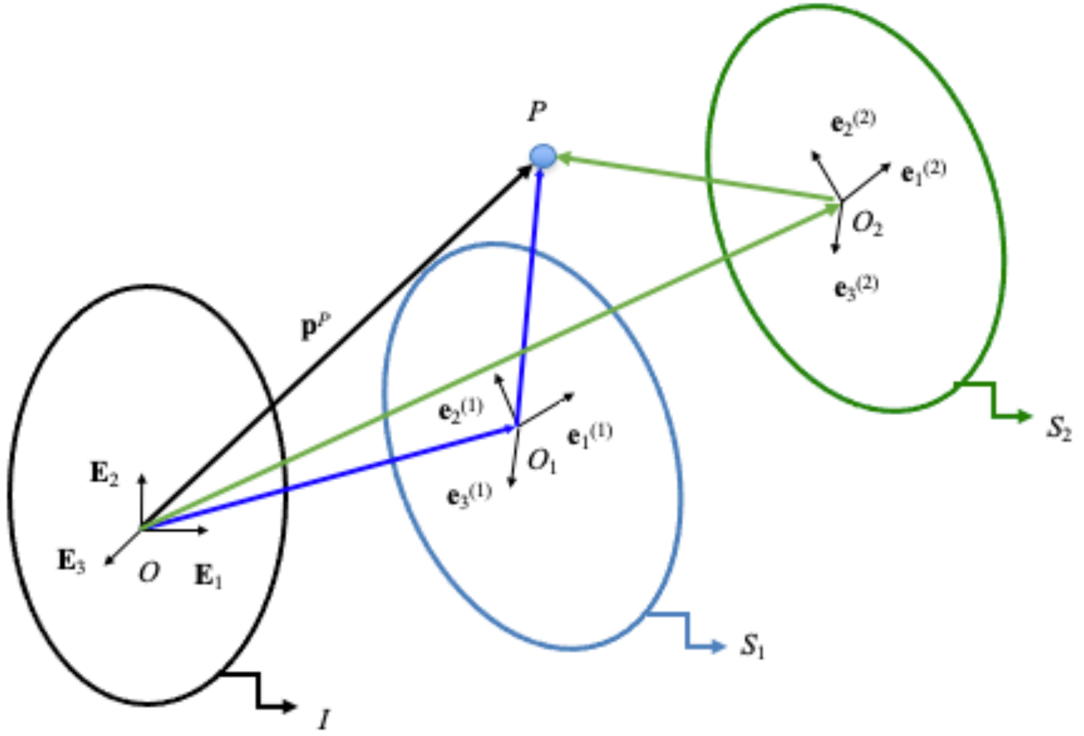


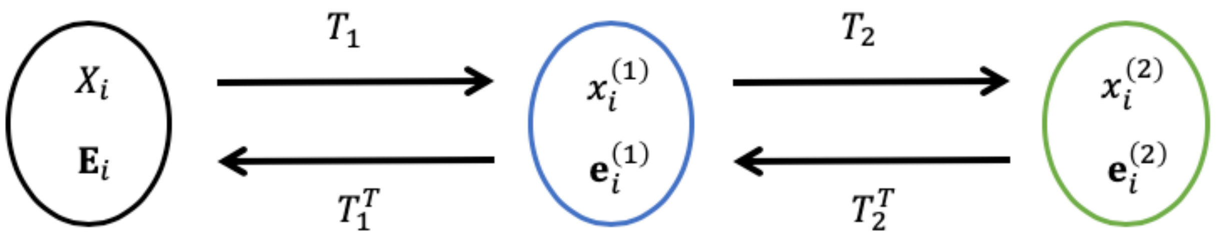
$$\mathbf{p}^P = \mathbf{p}^o + \mathbf{p}^{P/S}$$



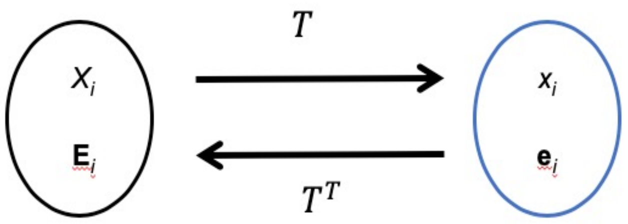
Referenciais Móveis

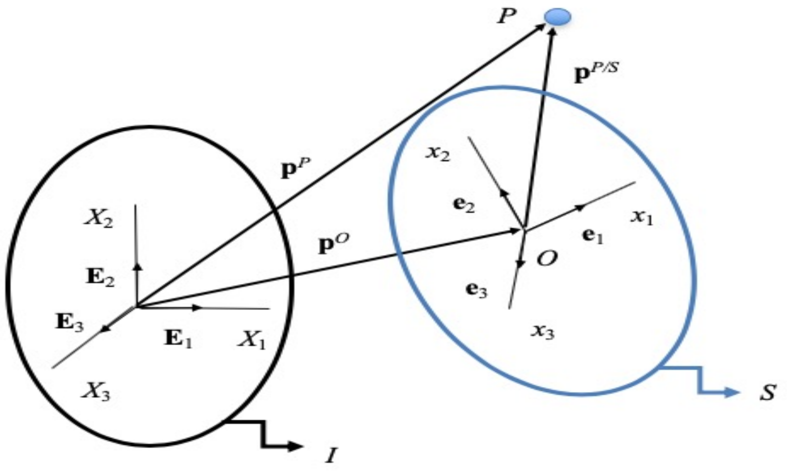
Aplicação recursiva em múltiplos referencias.





Referenciais Móveis





- **Notação**

Vetor posição da partícula *P* em relação ao referencial *I* e na base *S*

$${}^{I}_{S}\mathbf{p}^{P} \qquad {}_{S}\mathbf{p}^{P/I}$$

- **Referencial**

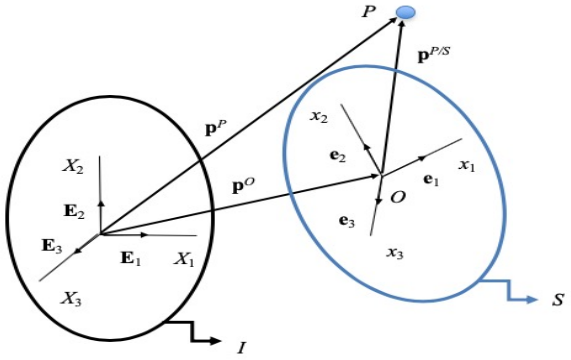
Em relação a *I*: absolutas

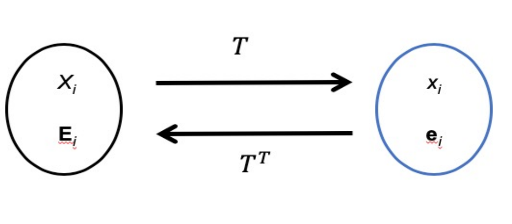
Em relação a *S*: relativas

- **Base (sistema coordenado)**

Questão puramente **geométrica (transformação de coordenadas)**

Referenciais Móveis





- Posição

$${}_I\mathbf{p}^P = {}_I\mathbf{p}^O + \mathbf{T}^T\,{}^S_S\mathbf{p}^P$$

$$\boxed{\mathbf{p}^P = \mathbf{p}^o + \mathbf{p}^{P/S}}$$

- Velocidade

$${}_I\mathbf{v}^P = \mathbf{v}^P = \frac{{}^I d\left({}_I\mathbf{p}^P\right)}{dt} = {}_I\dot{\mathbf{p}}^O + \mathbf{T}^T\,{}_S\dot{\mathbf{p}}^{P/S} + \dot{\mathbf{T}}^T\,{}_S\mathbf{p}^{P/S}$$

$$\mathbf{T}^T\,{}_S\dot{\mathbf{p}}^{P/S} = \mathbf{T}^T\,{}_S\mathbf{v}^{P/S} = {}_I\mathbf{v}^{P/S} = \mathbf{v}^{P/S}$$

$$\dot{\mathbf{T}}^T\,{}_S\mathbf{p}^{P/S} = \dot{\mathbf{T}}^T\mathbf{T}^T\,{}_I\mathbf{p}^{P/S} = \mathbf{T}\left(\mathbf{T}^T\dot{\mathbf{T}}^T\right)\mathbf{T}^T\,{}_I\mathbf{p}^{P/S}$$
$$= {}^I\boldsymbol{\omega}^S\times{}_I\mathbf{p}^{P/S} = {}^I\boldsymbol{\omega}^S\times\mathbf{p}^{P/S}$$

$${}_I\mathbf{v}^P = {}_I\mathbf{v}^O + \mathbf{T}^T\,{}_S\mathbf{v}^{P/S} + {}^I\boldsymbol{\omega}^S\times{}_I\mathbf{p}^{P/S}$$

$$\boxed{\mathbf{v}^P = \mathbf{v}^O + \mathbf{v}^{P/S} + {}^I\boldsymbol{\omega}^S\times\mathbf{p}^{P/S}}$$

$\mathbf{v}^P$ - velocidade de uma partícula *P*

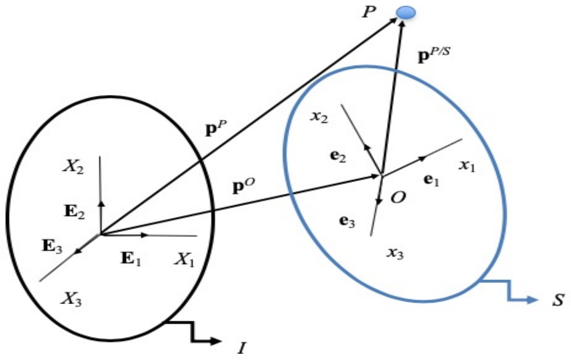
$\mathbf{v}^O$ - velocidade da origem do referencial móvel

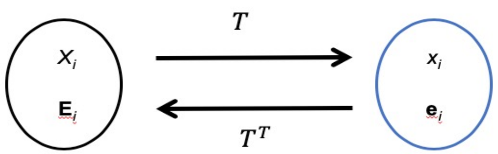
$\mathbf{v}^{P/S}$ - velocidade relativa da partícula em relação ao referencial móvel

${}^I\boldsymbol{\omega}^S\times\mathbf{p}^{P/S}$ - termo associado à variação dos vetores base

Referenciais Móveis

- Aceleração





$$ {}_I\mathbf{a}^P = {}_I\dot{\mathbf{v}}^P = \frac{{}^I d\left({}_I\mathbf{v}^P\right)}{dt} = {}_I\dot{\mathbf{v}}^O + \mathbf{T}^T\,{}_S\dot{\mathbf{v}}^{P/S} + \dot{\mathbf{T}}^T\,{}_S\mathbf{v}^{P/S} + {}^I\dot{\boldsymbol{\omega}}^S\times(\mathbf{T}^T\,{}_S\mathbf{p}^{P/S}) + {}^I\boldsymbol{\omega}^S\times(\dot{\mathbf{T}}^T\,{}_S\mathbf{p}^{P/S}) + {}^I\boldsymbol{\omega}^S\times(\mathbf{T}^T\,{}_S\dot{\mathbf{p}}^{P/S}) $$

$$ {}_I\dot{\mathbf{v}}^O = {}_I\mathbf{a}^O $$

$$ \mathbf{T}^T\,{}_S\dot{\mathbf{v}}^{P/S} = \mathbf{T}^T\,{}_S\mathbf{a}^{P/S} $$

$$ \dot{\mathbf{T}}^T\,{}_S\mathbf{v}^{P/S} = {}^I\boldsymbol{\omega}^S\times(\mathbf{T}^T\,{}_S\mathbf{v}^{P/S}) = {}^I\boldsymbol{\omega}^S\times\,{}_I\mathbf{v}^{P/S} $$

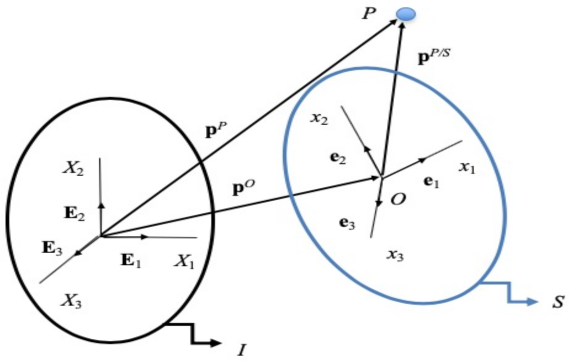
$$ {}^I\dot{\boldsymbol{\omega}}^S\times(\mathbf{T}^T\,{}_S\mathbf{p}^{P/S}) = {}^I\boldsymbol{\alpha}^S\times(\mathbf{T}^T\,{}_S\mathbf{p}^{P/S}) = {}^I\boldsymbol{\alpha}^S\times\,{}_I\mathbf{p}^{P/S} $$

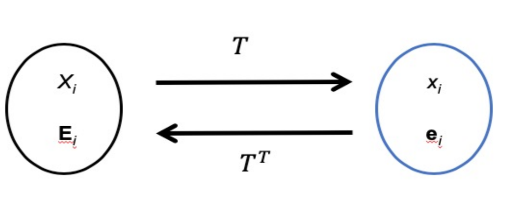
$$ {}^I\boldsymbol{\omega}^S\times(\dot{\mathbf{T}}^T\,{}_S\mathbf{p}^{P/S}) = {}^I\boldsymbol{\omega}^S\times({}^I\boldsymbol{\omega}^S\times(\mathbf{T}^T\,{}_S\mathbf{p}^{P/S})) $$

$$ = {}^I\boldsymbol{\omega}^S\times({}^I\boldsymbol{\omega}^S\times\,{}_I\mathbf{p}^{P/S}) $$

$$ {}^I\boldsymbol{\omega}^S\times(\mathbf{T}^T\,{}_S\dot{\mathbf{p}}^{P/S}) = {}^I\boldsymbol{\omega}^S\times(\mathbf{T}^T\,{}_S\mathbf{v}^{P/S}) = {}^I\boldsymbol{\omega}^S\times\,{}_I\mathbf{v}^{P/S} $$

Referenciais Móveis





- Aceleração

$$ {}_I\mathbf{a}^P = {}_I\dot{\mathbf{v}}^P = \frac{{}^I d\left({}_I\mathbf{v}^P\right)}{dt} = {}_I\dot{\mathbf{v}}^O + \mathbf{T}^T\,{}_S\dot{\mathbf{v}}^{P/S} + \dot{\mathbf{T}}^T\,{}_S\mathbf{v}^{P/S} + {}^I\dot{\boldsymbol{\omega}}^S\times(\mathbf{T}^T\,{}_S\mathbf{p}^{P/S}) + {}^I\boldsymbol{\omega}^S\times(\dot{\mathbf{T}}^T\,{}_S\mathbf{p}^{P/S}) + {}^I\boldsymbol{\omega}^S\times(\mathbf{T}^T\,{}_S\dot{\mathbf{p}}^{P/S}) $$

$$ {}_I\mathbf{a}^P = {}_I\mathbf{a}^O + \mathbf{T}^T\,{}_S\mathbf{a}^{P/S} + {}^I\boldsymbol{\alpha}^S\times(\mathbf{T}^T\,{}_S\mathbf{p}^{P/S}) + {}^I\boldsymbol{\omega}^S\times({}^I\boldsymbol{\omega}^S\times(\mathbf{T}^T\,{}_S\mathbf{p}^{P/S})) + 2\,{}^I\boldsymbol{\omega}^S\times(\mathbf{T}^T\,{}_S\mathbf{v}^{P/S}) $$

$$ \mathbf{a}^P = \mathbf{a}^O + \mathbf{a}^{P/S} + {}^I\boldsymbol{\alpha}^S\times\mathbf{p}^{P/S} + {}^I\boldsymbol{\omega}^S\times({}^I\boldsymbol{\omega}^S\times\mathbf{p}^{P/S}) + 2\,{}^I\boldsymbol{\omega}^S\times\mathbf{v}^{P/S} $$

$\mathbf{a}^P$ - aceleração da partícula P

$\mathbf{a}^O$ - aceleração da origem do referencial móvel

$\mathbf{a}^{P/S}$ - aceleração relativa de P em relação ao sistema móvel S

${}^I\boldsymbol{\alpha}^S\times\mathbf{p}^{P/S}$ - aceleração tangencial

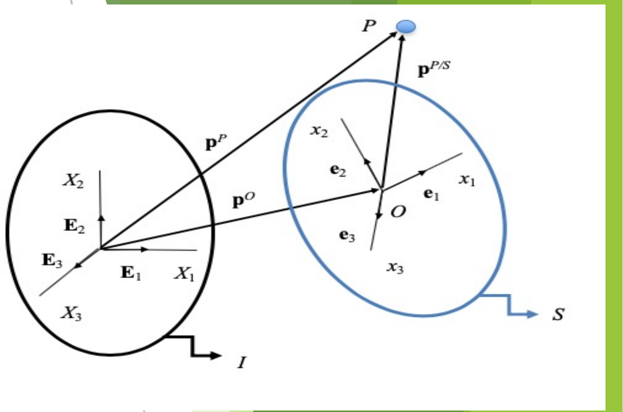
${}^I\boldsymbol{\omega}^S\times({}^I\boldsymbol{\omega}^S\times\mathbf{p}^{P/S})$- aceleração centrípeta

$2\,{}^I\boldsymbol{\omega}^S\times\mathbf{v}^{P/S}$ - aceleração de Coriolis, definida a partir da velocidade relativa

Interpretando os Referencias Móveis

- **Posição**

$$ {}_{S}\mathbf{p}^{P/S} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 $$



- **Velocidade**

$$ {}_{I}\mathbf{v}^{P} = \mathbf{v}^{P} = \frac{{}^{I}d(\mathbf{p}^{P})}{dt} = \frac{{}^{I}d(\mathbf{p}^{O})}{dt} + \frac{{}^{I}d({}_{S}\mathbf{p}^{P/S})}{dt} $$

$$ \mathbf{v}^{O} = \frac{d(\mathbf{p}^{O})}{dt} $$

$$ \frac{{}^{I}d({}_{S}\mathbf{p}^{P/S})}{dt} = (\dot{x}_1\mathbf{e}_1 + \dot{x}_2\mathbf{e}_2 + \dot{x}_3\mathbf{e}_3) + (x_1\dot{\mathbf{e}}_1 + x_2\dot{\mathbf{e}}_2 + x_3\dot{\mathbf{e}}_3) $$

$$ \mathbf{v}^{P/S} = \dot{x}_1\mathbf{e}_1 + \dot{x}_2\mathbf{e}_2 + \dot{x}_3\mathbf{e}_3 $$

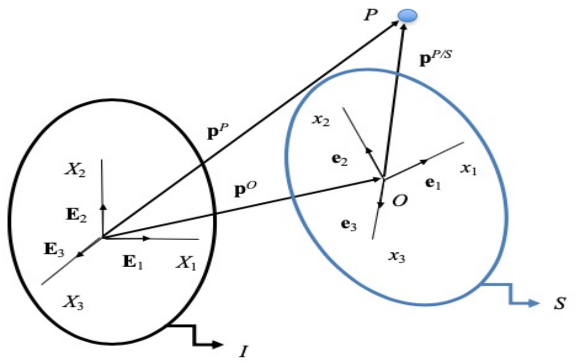
$$ x_1\dot{\mathbf{e}}_1 + x_2\dot{\mathbf{e}}_2 + x_3\dot{\mathbf{e}}_3 = {}^{I}\boldsymbol{\omega}^{S}\times x_1\mathbf{e}_1 + {}^{I}\boldsymbol{\omega}^{S}\times x_2\mathbf{e}_2 + {}^{I}\boldsymbol{\omega}^{S}\times x_3\mathbf{e}_3 = {}^{I}\boldsymbol{\omega}^{S}\times\mathbf{p}^{P/S} $$

$$ \boxed{\mathbf{v}^{P} = \mathbf{v}^{O} + \mathbf{v}^{P/S} + {}^{I}\boldsymbol{\omega}^{S}\times\mathbf{p}^{P/S}} $$

Interpretando os Referencias Móveis

- **Aceleração**

$$ {}^{I}\mathbf{a}^{P} = \mathbf{a}^{P} = \frac{{}^{I}d(\mathbf{v}^{P})}{dt} = \frac{{}^{I}d}{dt}(\mathbf{v}^{O} + \mathbf{v}^{P/S} + {}^{I}\boldsymbol{\omega}^{S}\times\mathbf{p}^{P/S}) $$



$$ \frac{{}^{I}d(\mathbf{v}^{P/S})}{dt} = \frac{{}^{I}d}{dt}(\dot{x}_1\mathbf{e}_1 + \dot{x}_2\mathbf{e}_2 + \dot{x}_3\mathbf{e}_3) = (\ddot{x}_1\mathbf{e}_1 + \ddot{x}_2\mathbf{e}_2 + \ddot{x}_3\mathbf{e}_3) + (\dot{x}_1\dot{\mathbf{e}}_1 + \dot{x}_2\dot{\mathbf{e}}_2 + \dot{x}_3\dot{\mathbf{e}}_3) $$

$$ \mathbf{a}^{P/S} = \ddot{x}_1\mathbf{e}_1 + \ddot{x}_2\mathbf{e}_2 + \ddot{x}_3\mathbf{e}_3 $$

$$ \dot{x}_1\dot{\mathbf{e}}_1 + \dot{x}_2\dot{\mathbf{e}}_2 + \dot{x}_3\dot{\mathbf{e}}_3 = \dot{x}_1({}^{I}\boldsymbol{\omega}^{S}\times\mathbf{e}_1) + \dot{x}_2({}^{I}\boldsymbol{\omega}^{S}\times\mathbf{e}_2) + \dot{x}_3({}^{I}\boldsymbol{\omega}^{S}\times\mathbf{e}_3) = {}^{I}\boldsymbol{\omega}^{S}\times(\dot{x}_1\mathbf{e}_1 + \dot{x}_2\mathbf{e}_2 + \dot{x}_3\mathbf{e}_3) $$
$$ = {}^{I}\boldsymbol{\omega}^{S}\times\mathbf{v}^{P/S} $$

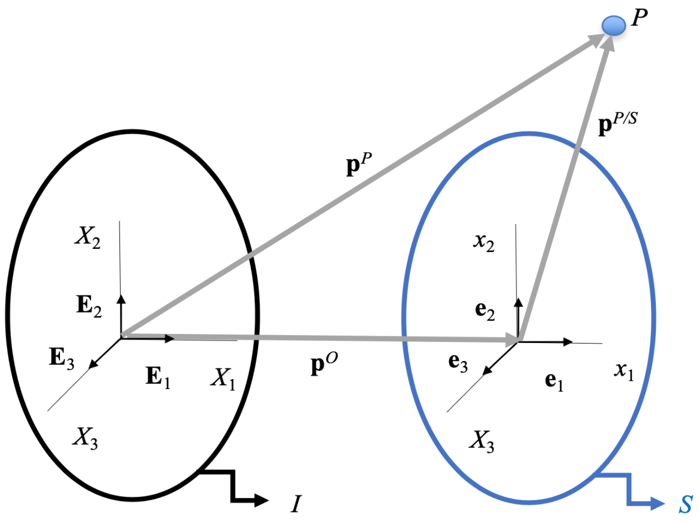
$$ \frac{{}^{I}d}{dt}({}^{I}\boldsymbol{\omega}^{S}\times\mathbf{p}^{P/S}) = \frac{{}^{I}d\ {}^{I}\boldsymbol{\omega}^{S}}{dt}\times\mathbf{p}^{P/S} + {}^{I}\boldsymbol{\omega}^{S}\times\frac{{}^{I}d\mathbf{p}^{P/S}}{dt} $$

$$ {}^{I}\boldsymbol{\omega}^{S}\times\frac{{}^{I}d\mathbf{p}^{P/S}}{dt} = {}^{I}\boldsymbol{\omega}^{S}\times(\dot{x}_1\mathbf{e}_1 + \dot{x}_2\mathbf{e}_2 + \dot{x}_3\mathbf{e}_3) + (x_1\dot{\mathbf{e}}_1 + x_2\dot{\mathbf{e}}_2 + x_3\dot{\mathbf{e}}_3) = {}^{I}\boldsymbol{\omega}^{S}\times\mathbf{v}^{P/S} + {}^{I}\boldsymbol{\omega}^{S}\times({}^{I}\boldsymbol{\omega}^{S}\times\mathbf{p}^{P/S}) $$

$$ \frac{{}^{I}d\ {}^{I}\boldsymbol{\omega}^{S}}{dt}\times\mathbf{p}^{P/S} = {}^{I}\boldsymbol{\alpha}^{S}\times\mathbf{p}^{P/S} $$

$$ \mathbf{a}^{P} = \mathbf{a}^{O} + \mathbf{a}^{P/S} + {}^{I}\boldsymbol{\alpha}^{S}\times\mathbf{p}^{P/S} + {}^{I}\boldsymbol{\omega}^{S}\times({}^{I}\boldsymbol{\omega}^{S}\times\mathbf{p}^{P/S}) + 2\,{}^{I}\boldsymbol{\omega}^{S}\times\mathbf{v}^{P/S} $$

Translação



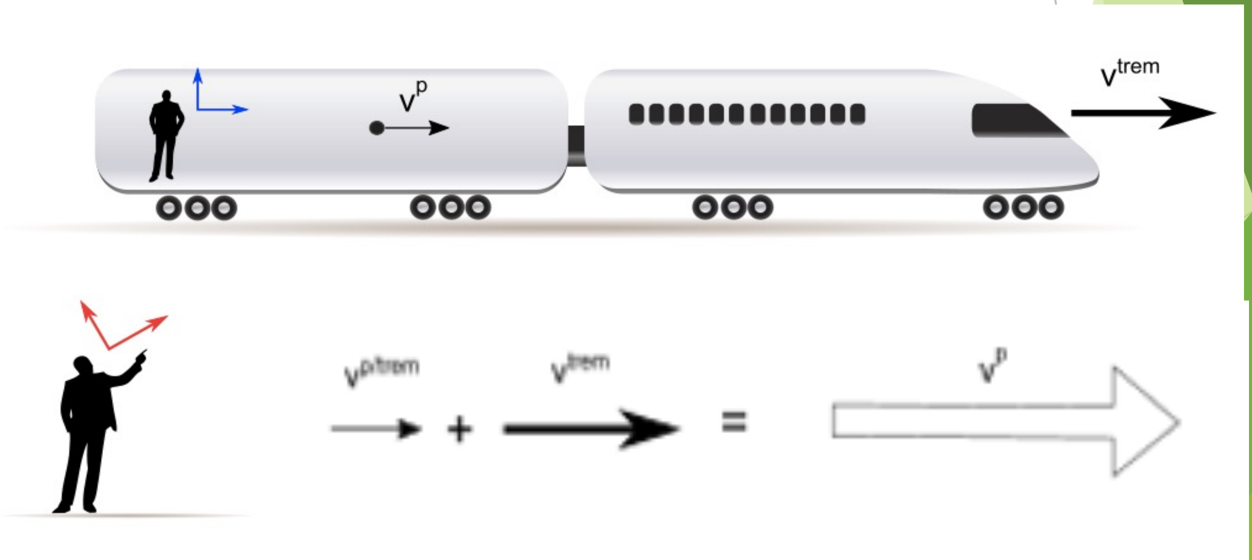
$$\mathbf{T} = \mathbf{T}^T = \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dot{\mathbf{I}} = 0$$

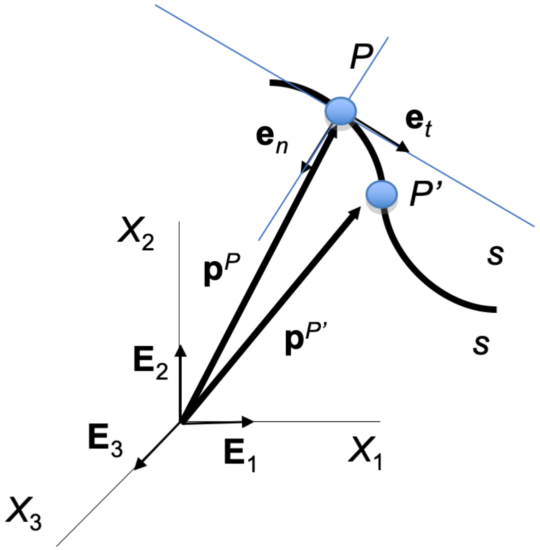
$$\mathbf{p}^P = \mathbf{p}^O + \mathbf{I}\,{}_S\mathbf{p}^{P/S}$$

$$\mathbf{v}^P = \mathbf{v}^O + \mathbf{I}\,{}_S\mathbf{v}^{P/S} = \mathbf{v}^O + \mathbf{v}^{P/S}$$

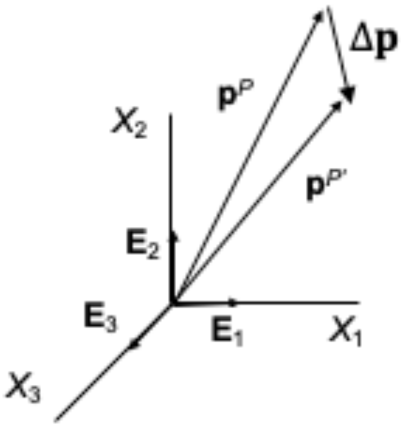
$$\mathbf{a}^P = \mathbf{a}^O + \mathbf{I}\,{}_S\mathbf{a}^{P/S} = \mathbf{a}^O + \mathbf{a}^{P/S}$$



Sistema Normal-Tangente: Velocidade



$$\mathbf{v}^P = \frac{d\mathbf{p}^P}{dt} = \dot{\mathbf{p}}^P = \lim_{\Delta t \to 0} \frac{\mathbf{p}^{P'} - \mathbf{p}^P}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{p}}{\Delta t}$$

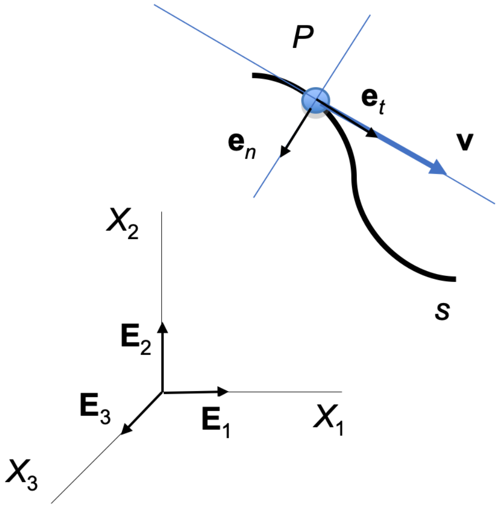


$$\mathbf{v}^P = \frac{d\mathbf{p}^P}{dt} = \frac{d\mathbf{p}^P}{ds}\frac{ds}{dt}$$

$$\frac{ds}{dt} = |\mathbf{v}^P| = v^P$$

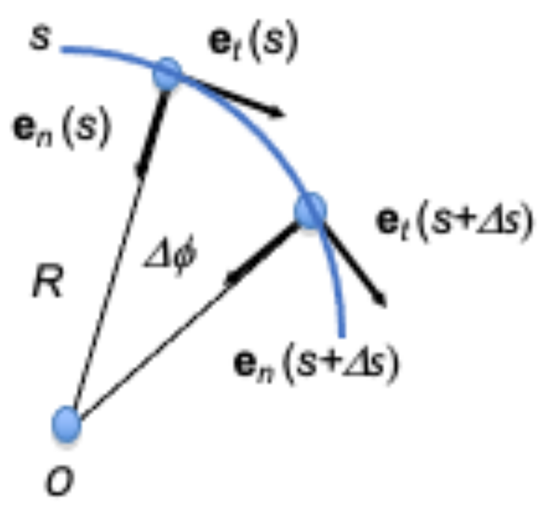
$$\frac{d\mathbf{p}}{ds} = \lim_{\Delta s \to 0} \frac{\Delta \mathbf{p}}{\Delta s} = \mathbf{e}_t$$

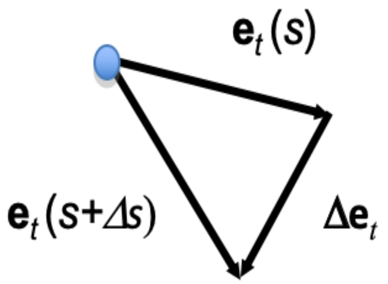
$$\mathbf{v}^P = v^P \mathbf{e}_t = \frac{ds}{dt}\mathbf{e}_t$$

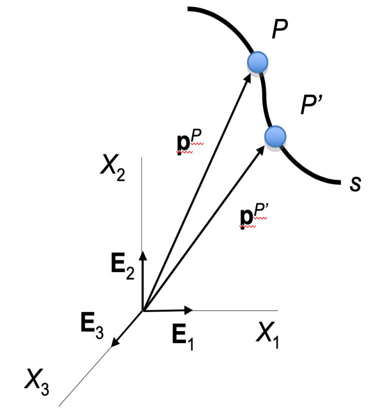


Sistema Normal-Tangente: Aceleração

$$\mathbf{a}^P = \frac{d\mathbf{v}^P}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\mathbf{e}_t\right) = \frac{d^2s}{dt^2}\mathbf{e}_t + \frac{ds}{dt}\dot{\mathbf{e}}_t$$







$$\dot{\mathbf{e}}_t = \frac{d\mathbf{e}_t}{ds}\frac{ds}{dt}$$

$$\frac{d\mathbf{e}_t}{ds} = \lim_{\Delta s \to 0}\left(\frac{\mathbf{e}_t(s+\Delta s) - \mathbf{e}_t(s)}{\Delta s}\right) = \lim_{\Delta s \to 0}\frac{\Delta \mathbf{e}_t}{\Delta s}$$

$$\Delta e_t = |\Delta \mathbf{e}_t| \approx |\mathbf{e}_t|\Delta\phi = \Delta\phi$$

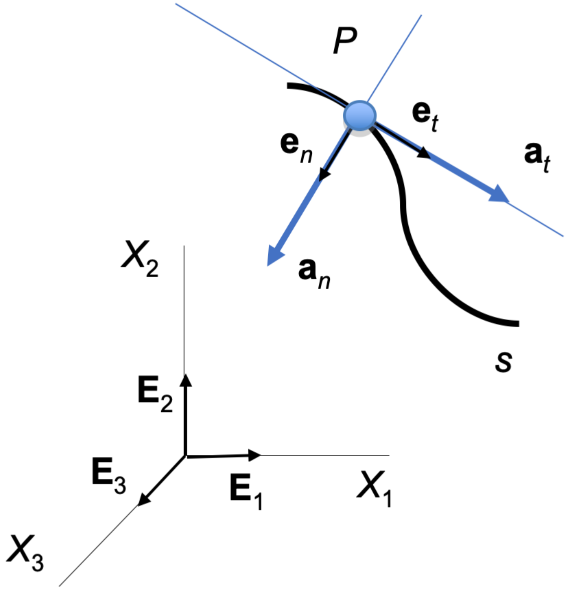
$$\Delta s \approx R\Delta\phi$$

$$\frac{de_t}{ds} = \lim_{\Delta s \to 0}\frac{\Delta e_t}{\Delta s} = \lim_{\Delta s \to 0}\frac{1}{\Delta s}\left(\frac{\Delta s}{R}\right) = \frac{1}{R}$$

$$\frac{d\mathbf{e}_t}{ds} = \frac{1}{R}\mathbf{e}_n$$

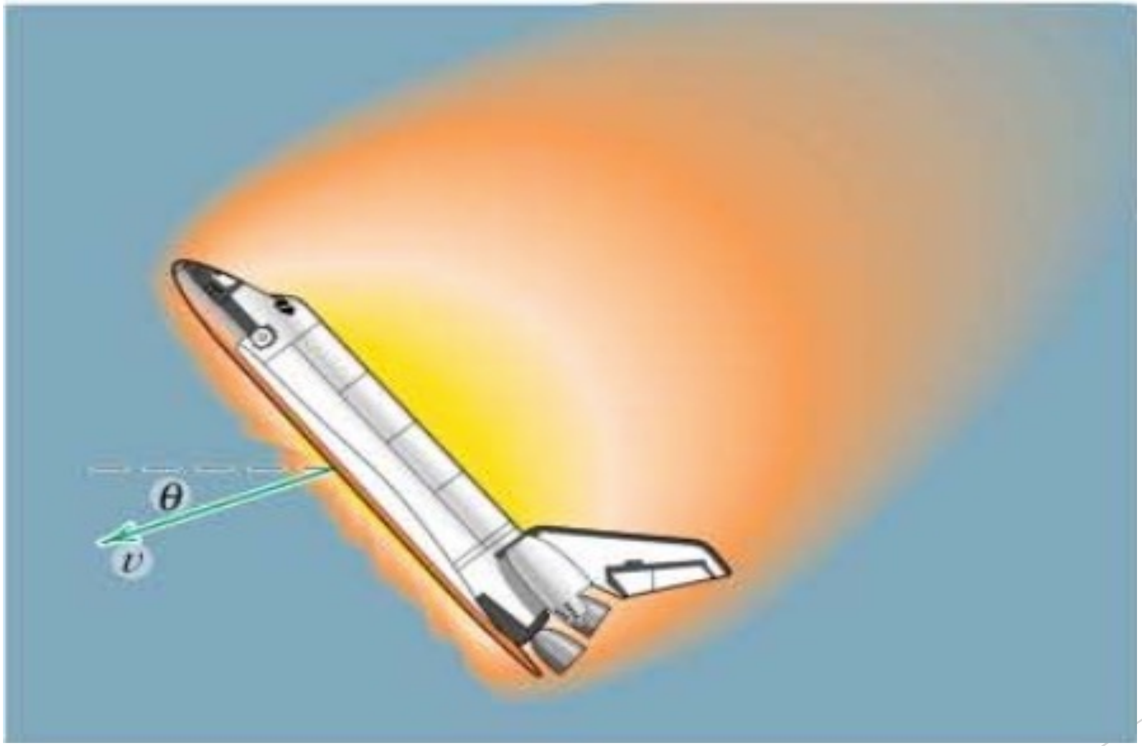
$$\mathbf{a}^P = \frac{d^2s}{dt^2}\mathbf{e}_t + \frac{1}{R}\left(\frac{ds}{dt}\right)^2\mathbf{e}_n$$

$$\mathbf{a}^P = a_t\mathbf{e}_t + a_n\mathbf{e}_n$$



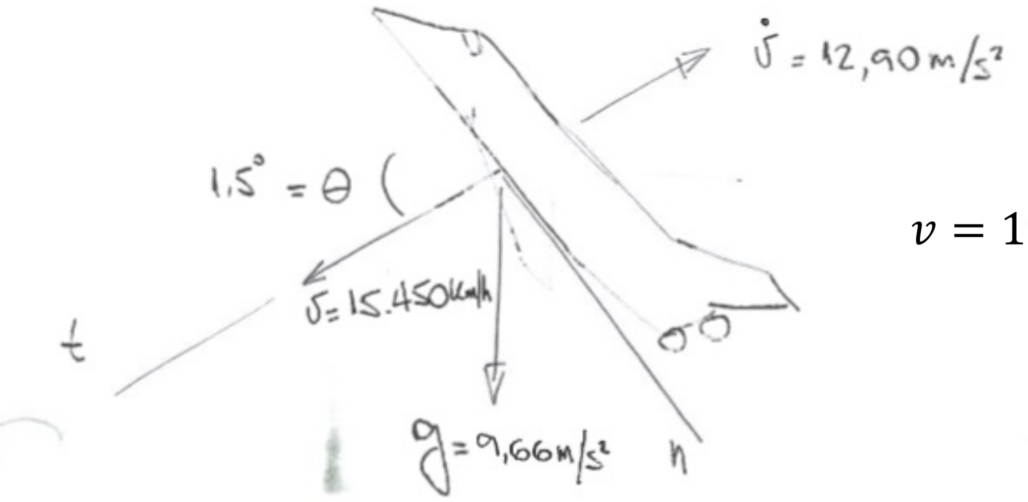
Reentrada de um Ônibus Espacial

Em um certo ponto na reentrada do ônibus espacial na atmosfera da Terra, a aceleração total pode ser representada por duas componentes: a aceleração da gravidade, 9,66 m/s^2 (na distância da Terra); e a resistência do ar da atmosfera se opondo ao movimento, 12,90 m/s^2. O ônibus espacial está a uma altitude de 48,2 km e sua velocidade orbital é reduzida de 28.300 km/h para 15.450 km/h na direção $\theta = 1{,}5°$. Para esse instante, avalie o raio de curvatura R da trajetória e a taxa com a qual o módulo da velocidade está mudando.



Reentrada de um Ônibus Espacial

Considerando o ônibus como uma partícula:



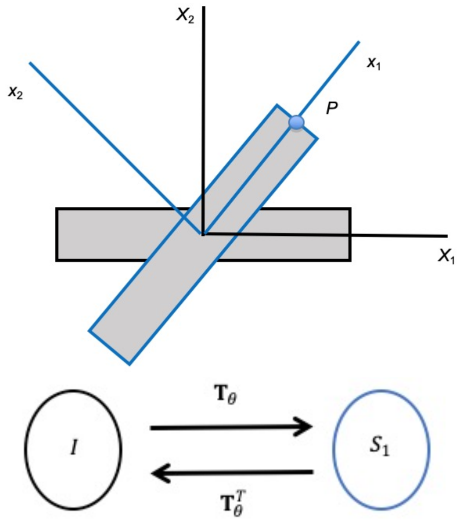
$$v = 15.450\,\frac{\text{km}}{\text{h}}\left(\frac{1000\text{m}}{1\text{km}}\frac{1\text{h}}{3600\text{s}}\right) = 4.291{,}67\,\frac{\text{m}}{\text{s}}$$

$$a_t = g\ \text{sen}\ \theta - \dot{v} = -12{,}65\ \ \text{m/s}^2$$

$$a_n = g\cos\theta = 9{,}657\ \ \text{m/s}^2$$

$$R = \frac{v^2}{a_n} = \frac{(4.291{,}67)^2}{9{,}657} = 1.907\ \text{km}$$

Movimento de Rotação



$$\mathbf{v}^P = \mathbf{v}^O + \mathbf{v}^{P/S} + {}^I\boldsymbol{\omega}^S \times \mathbf{p}^{P/S}$$

0 0

$$_{S_1}\mathbf{v}^P = {}_{S_1}^{I}\boldsymbol{\omega}^S \times {}_{S_1}\mathbf{p}^P = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \dot{\theta} \\ R & 0 & 0 \end{vmatrix} = \begin{Bmatrix} 0 \\ R\dot{\theta} \\ 0 \end{Bmatrix}$$

$$\mathbf{a}^P = \mathbf{a}^O + \mathbf{a}^{P/S} + \boldsymbol{\alpha}^S \times \mathbf{p}^{P/S} + \boldsymbol{\omega}^S \times \left(\boldsymbol{\omega}^S \times \mathbf{p}^{P/S}\right) + 2\boldsymbol{\omega}^S \times \mathbf{v}^{P/S} = a_t \mathbf{e}_t + a_n \mathbf{e}_n$$

0 0 0

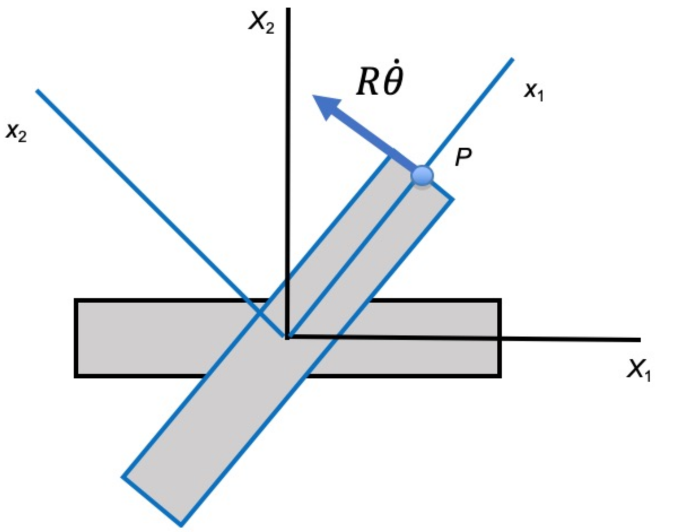
$$_{S_1}^{I}\boldsymbol{\alpha}^{S_1} \times {}_{S_1}\mathbf{p}^{P/S_1} = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \ddot{\theta} \\ R & 0 & 0 \end{vmatrix} = \begin{Bmatrix} 0 \\ R\ddot{\theta} \\ 0 \end{Bmatrix}$$

$$_{S_1}^{I}\boldsymbol{\omega}^{S_1} \times \left({}_{S_1}^{I}\boldsymbol{\omega}^{S_1} \times {}_{S_1}\mathbf{p}^{P/S_1}\right) = {}_{S_1}^{I}\boldsymbol{\omega}^{S_1} \times \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \dot{\theta} \\ R & 0 & 0 \end{vmatrix} = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \dot{\theta} \\ 0 & R\dot{\theta} & 0 \end{vmatrix} = \begin{Bmatrix} -R\dot{\theta}^2 \\ 0 \\ 0 \end{Bmatrix}$$

Movimento de Rotação

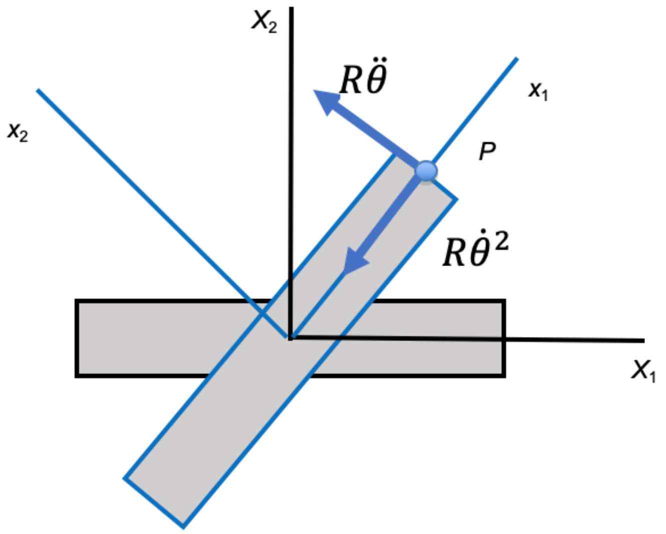
$${}_{S_1}\mathbf{v}^P = \begin{Bmatrix} 0 \\ R\dot{\theta} \\ 0 \end{Bmatrix}$$

$${}_I\mathbf{v}^P = \mathbf{T}_\theta^T\,{}_{S_1}\mathbf{v}^P = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ R\dot{\theta} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -R\dot{\theta}\, s\theta \\ R\dot{\theta}\, c\theta \\ 0 \end{Bmatrix}$$



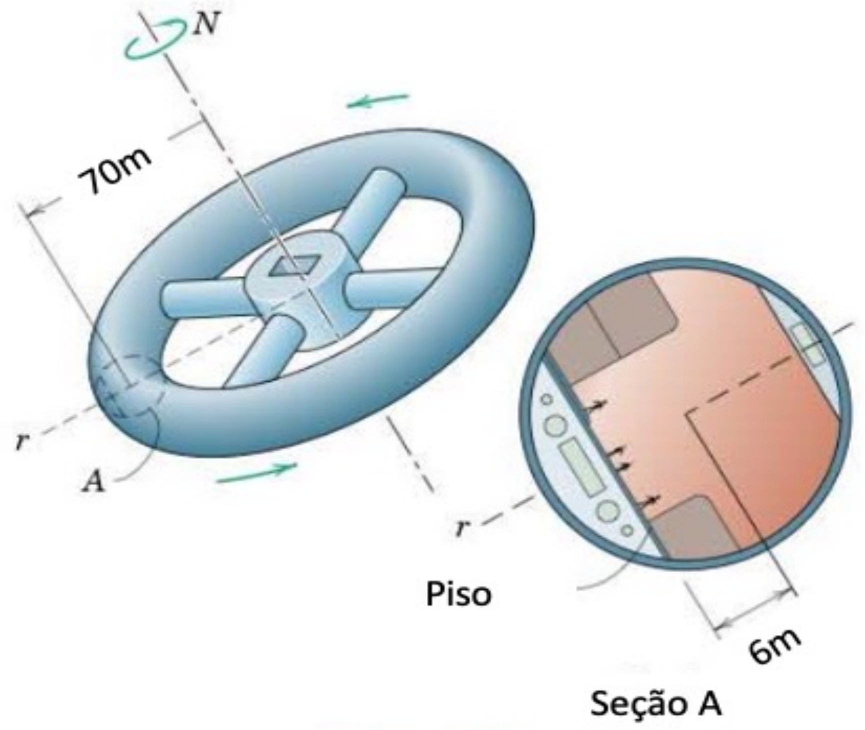
$${}_{S_1}^{I}\mathbf{a}^P = \begin{Bmatrix} -R\dot{\theta}^2 \\ R\ddot{\theta} \\ 0 \end{Bmatrix}$$

$${}_I^I\mathbf{a}^P = \mathbf{T}_\theta^T\,{}_{S_1}\mathbf{a}^P = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -R\dot{\theta}^2 \\ R\ddot{\theta} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -R\dot{\theta}^2\, c\theta - R\ddot{\theta}\, s\theta \\ -R\dot{\theta}^2\, s\theta + R\ddot{\theta}\, c\theta \\ 0 \end{Bmatrix}$$



Emulando a gravidade em uma estação espacial

Uma estação espacial que vai orbitar a Terra em uma trajetória circular consiste de um toro com uma seção transversal circular mostrada. O raio do toro é de aproximadamente 70m enquanto o espaço útil dentro do toro é mostrado na seção A, onde o nível térreo está a 6 m do centro da seção. Calcule a velocidade angular necessária para emular a gravidade padrão na superfície da terra (9,8 m/s^2).



Emulando a gravidade

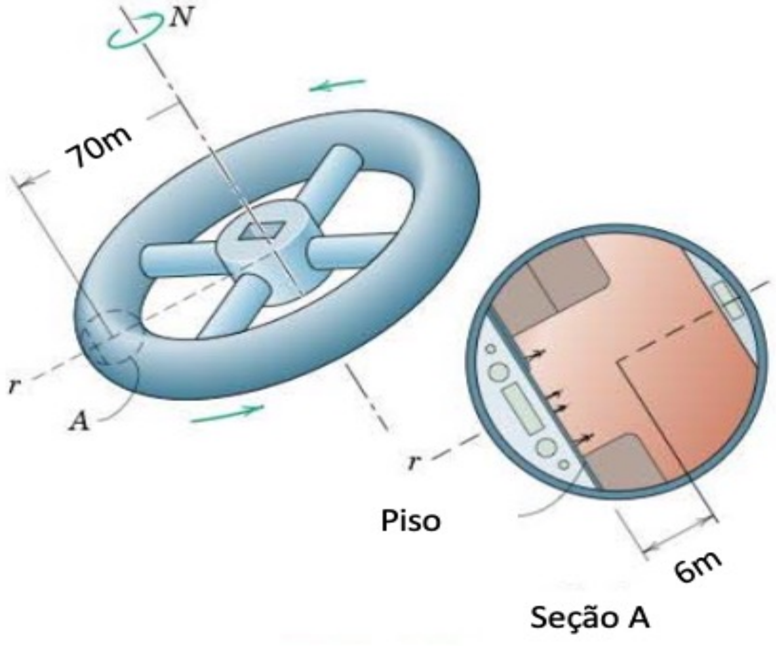
$a_n = r\dot{\theta}^2 = 9{,}8 \ \text{m/s}^2$

onde a distância $r = 70 + 6 = 76\text{m}$

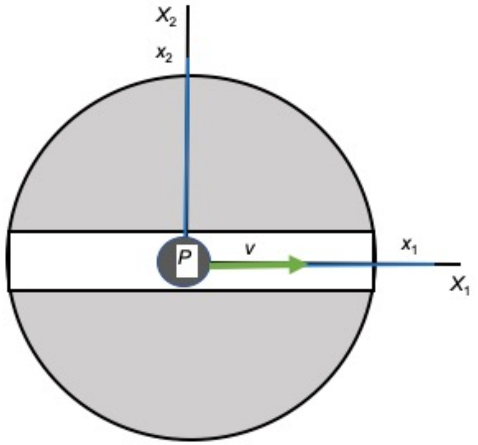
$\dot{\theta}^2 = \frac{a_n}{r} = \frac{9{,}8}{76} = 0{,}13 \ \text{rad/s}$

$\dot{\theta} = 0{,}36 \frac{\text{rad}}{\text{s}} \left(\frac{1}{2\pi \ \text{rad}} \frac{60s}{\text{min}} \right) = 3{,}43 \ \text{rpm}$

Desta forma, a rotação da estação emula a gravidade.



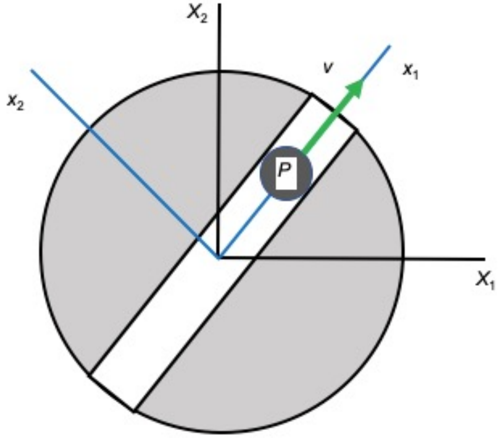
Movimento de Rotação Combinado
Movimento Relativo



$$\mathbf{v}^P = \mathbf{v}^O + \mathbf{v}^{P/S} + {}^I\boldsymbol{\omega}^S \times \mathbf{p}^{P/S}$$

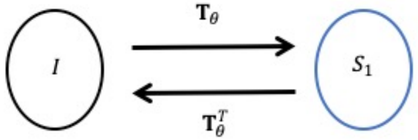
0

$$_{S_1}\mathbf{v}^{P/S} = \begin{Bmatrix} v \\ 0 \\ 0 \end{Bmatrix}$$



$$_{S_1}^{I}\boldsymbol{\omega}^S \times {}_{S_1}\mathbf{p}^P = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \dot{\theta} \\ R & 0 & 0 \end{vmatrix} = \begin{Bmatrix} 0 \\ R\dot{\theta} \\ 0 \end{Bmatrix}$$

$$_{S_1}\mathbf{v}^P = \begin{Bmatrix} v \\ R\dot{\theta} \\ 0 \end{Bmatrix}$$



$$_I\mathbf{v}^P = \mathbf{T}_\theta^T\, {}_{S_1}\mathbf{v}^P = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v \\ R\dot{\theta} \\ 0 \end{Bmatrix} = \begin{Bmatrix} vc\theta - R\dot{\theta}\, s\theta \\ vs\theta + R\dot{\theta}\, c\theta \\ 0 \end{Bmatrix}$$

Movimento de Rotação Combinado
Movimento Relativo

$$\mathbf{a}^P = \mathbf{a}^O + \mathbf{a}^{P/S} + \boldsymbol{\alpha}^S \times \mathbf{p}^{P/S} + \boldsymbol{\omega}^S \times (\boldsymbol{\omega}^S \times \mathbf{p}^{P/S}) + 2\boldsymbol{\omega}^S \times \mathbf{v}^{P/S}$$

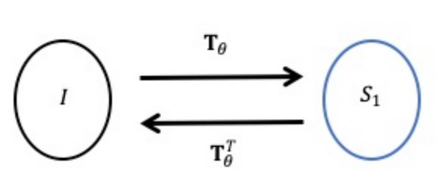
0 0 0

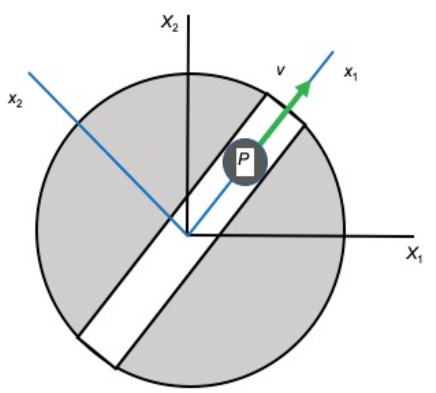
$${}_{S_1}^{I}\boldsymbol{\omega}^{S_1} \times ({}_{S_1}^{I}\boldsymbol{\omega}^{S_1} \times {}_{S_1}\mathbf{p}^{P/S_1}) = {}_{S_1}^{I}\boldsymbol{\omega}^{S_1} \times \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \dot{\theta} \\ R & 0 & 0 \end{vmatrix} = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \dot{\theta} \\ 0 & R\dot{\theta} & 0 \end{vmatrix} = \begin{Bmatrix} -R\dot{\theta}^2 \\ 0 \\ 0 \end{Bmatrix}$$

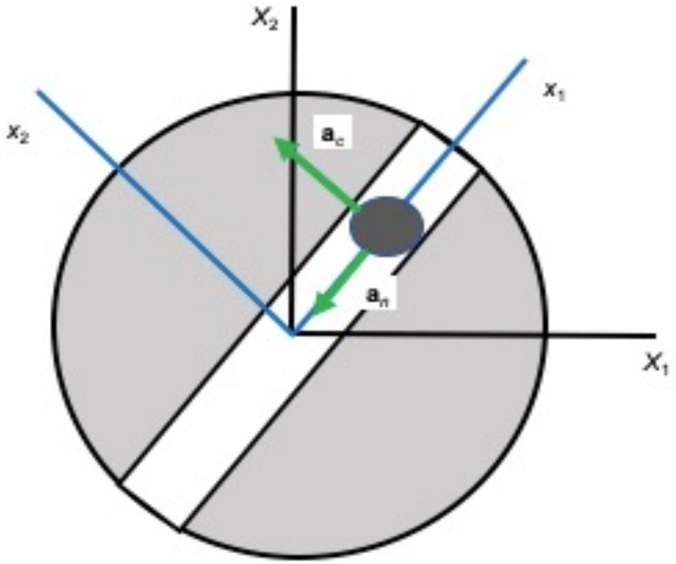
$$2\,{}_{S_1}^{I}\boldsymbol{\omega}^{S_1} \times {}_{S_1}\mathbf{v}^{P/S_1} = 2\begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \dot{\theta} \\ v & 0 & 0 \end{vmatrix} = \begin{Bmatrix} 0 \\ 2v\dot{\theta} \\ 0 \end{Bmatrix}$$

$${}_{S_1}^{I}\mathbf{a}^P = \begin{Bmatrix} -R\dot{\theta}^2 \\ 2v\dot{\theta} \\ 0 \end{Bmatrix}$$

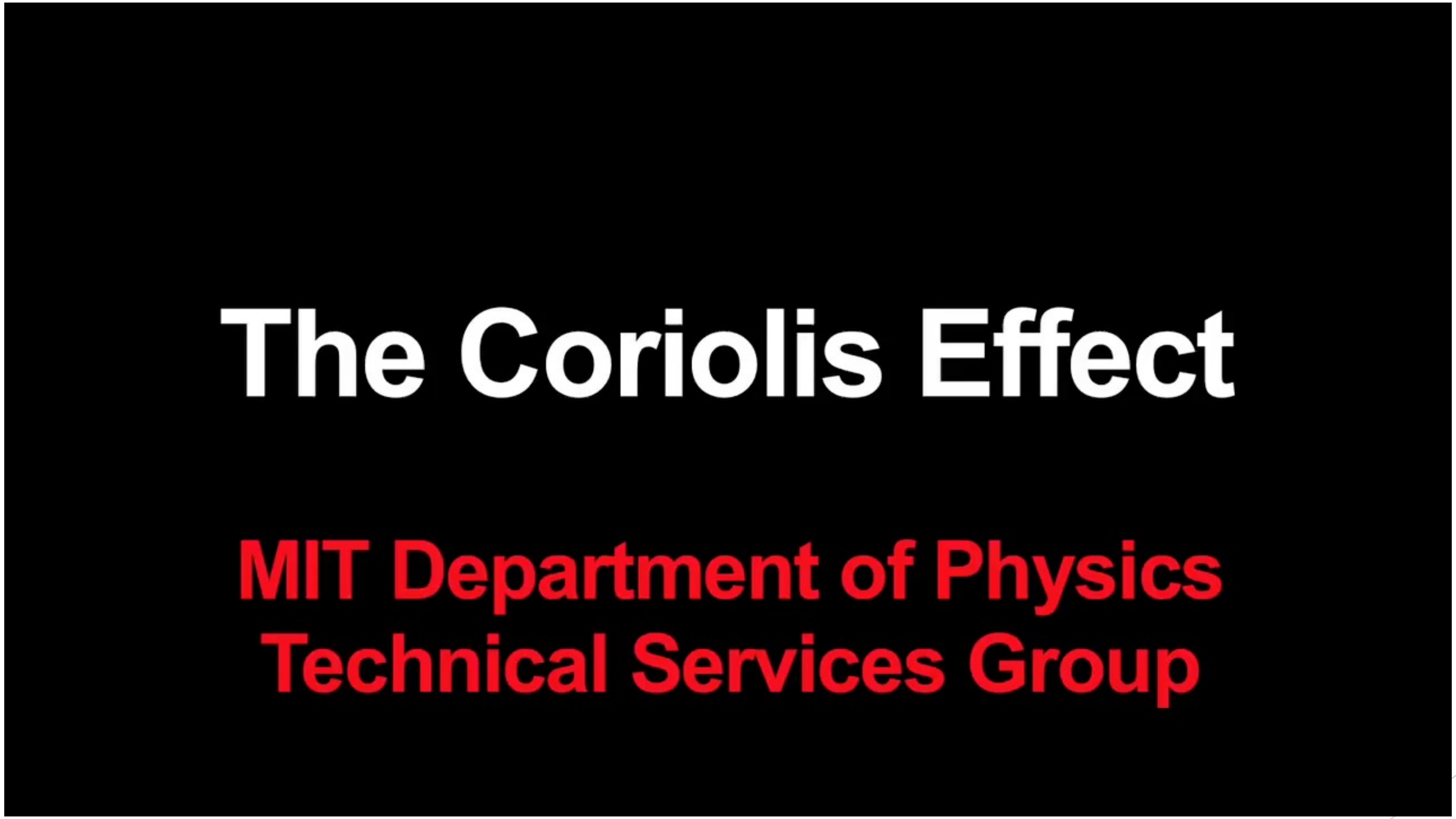
$${}_{I}^{I}\mathbf{a}^P = \mathbf{T}_\theta^T\,{}_{S_1}\mathbf{a}^P = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -R\dot{\theta}^2 \\ v\dot{\theta} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -R\dot{\theta}^2\, c\theta - 2v\dot{\theta}\, s\theta \\ -R\dot{\theta}^2\, s\theta + 2v\dot{\theta}\, c\theta \\ 0 \end{Bmatrix}$$

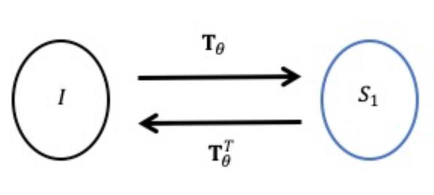


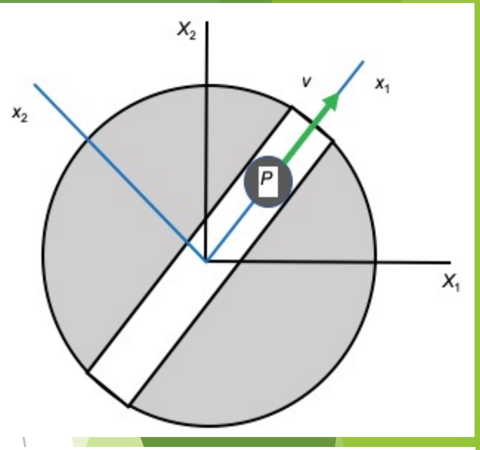


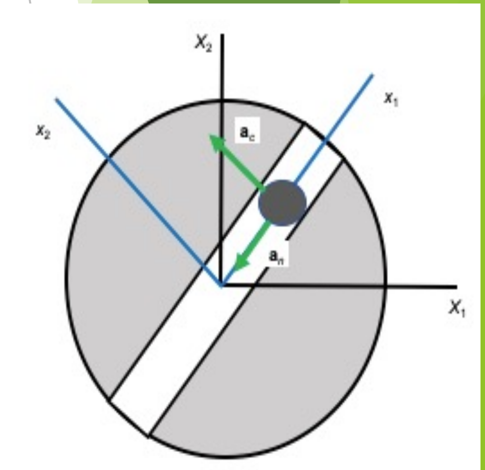


Movimento de Rotação Combinado
Movimento Relativo

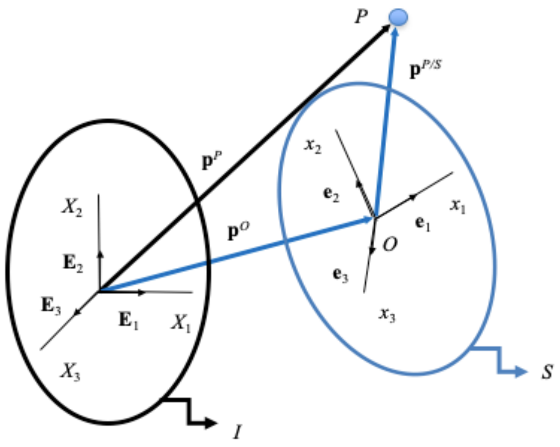








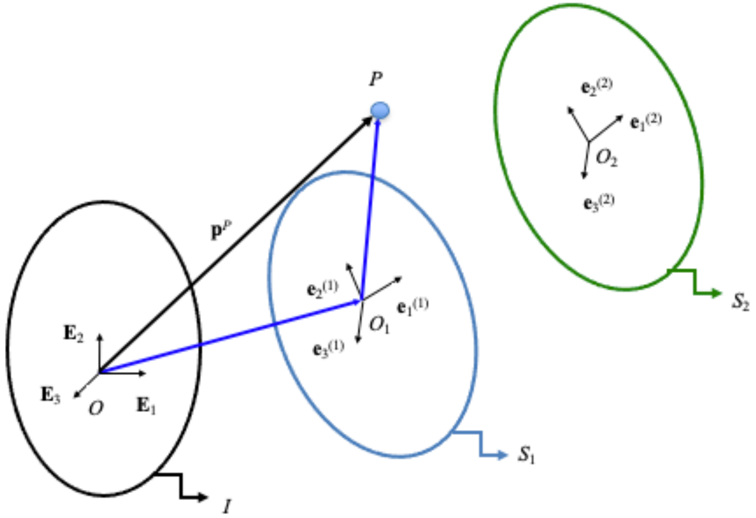
Velocidade Angular



$$\frac{{}^{I}d\mathbf{u}}{dt} = \frac{{}^{S}d\mathbf{u}}{dt} + {}^{I}\boldsymbol{\omega}^{S} \times \mathbf{u}$$

$$\frac{{}^{I}d\mathbf{u}}{dt} = \frac{{}^{S}d\mathbf{u}}{dt} + {}^{I}\boldsymbol{\omega}^{S} \times \mathbf{u} = \left[\frac{{}^{I}d\mathbf{u}}{dt} + {}^{S}\boldsymbol{\omega}^{I} \times \mathbf{u}\right] + {}^{I}\boldsymbol{\omega}^{S} \times \mathbf{u}$$

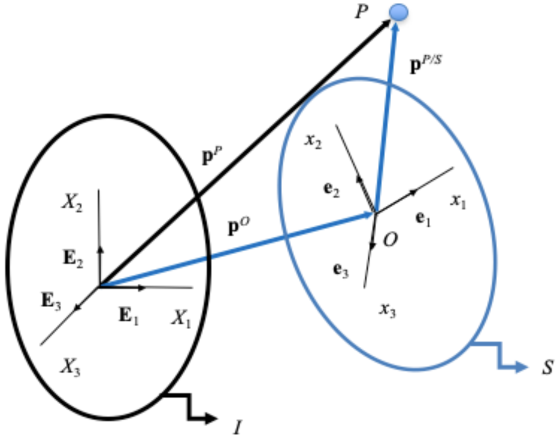
$$\left[{}^{S}\boldsymbol{\omega}^{I} + {}^{I}\boldsymbol{\omega}^{S}\right] \times \mathbf{u} = 0 \Rightarrow {}^{S}\boldsymbol{\omega}^{I} = -\,{}^{I}\boldsymbol{\omega}^{S}$$



$${}^{I}\boldsymbol{\omega}^{S_2} = {}^{I}\boldsymbol{\omega}^{S_1} + {}^{S_1}\boldsymbol{\omega}^{S_2}$$

$${}^{I}\boldsymbol{\omega}^{S_N} = {}^{I}\boldsymbol{\omega}^{S_1} + {}^{S_1}\boldsymbol{\omega}^{S_2} + {}^{S_2}\boldsymbol{\omega}^{S_3} + \cdots + {}^{S_{N-1}}\boldsymbol{\omega}^{S_N}$$

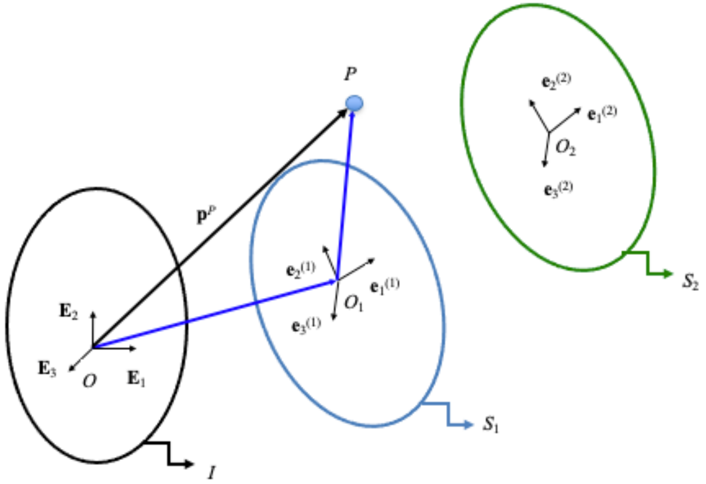
Aceleração Angular



$$^{I}\boldsymbol{\alpha}^{S} = \frac{^{I}d\ ^{I}\boldsymbol{\omega}^{S}}{dt}$$

$$^{I}\boldsymbol{\alpha}^{S} = \frac{^{I}d\ ^{I}\boldsymbol{\omega}^{S}}{dt} = \frac{^{S}d\ ^{I}\boldsymbol{\omega}^{S}}{dt} + {}^{S}\boldsymbol{\omega}^{I} \times {}^{I}\boldsymbol{\omega}^{S} = -\frac{^{S}d\ ^{S}\boldsymbol{\omega}^{I}}{dt} = -\,^{S}\boldsymbol{\alpha}^{I}$$

$$^{I}\boldsymbol{\alpha}^{S} = -\,^{S}\boldsymbol{\alpha}^{I}$$



$$^{I}\boldsymbol{\omega}^{S_2} = {}^{I}\boldsymbol{\omega}^{S_1} + {}^{S_1}\boldsymbol{\omega}^{S_2}$$

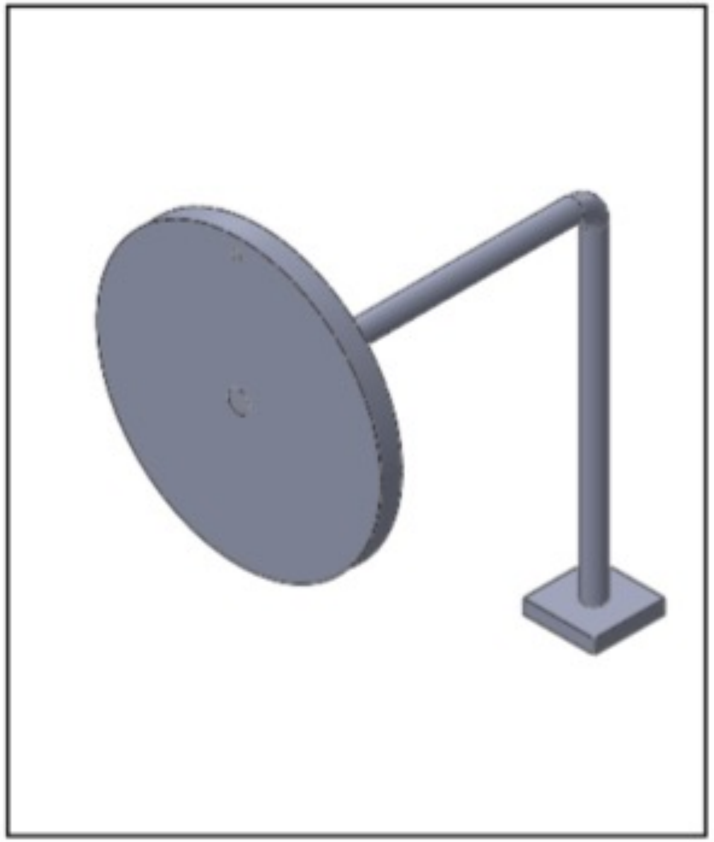
$$\frac{^{I}d\ ^{I}\boldsymbol{\omega}^{S_2}}{dt} = \frac{^{I}d}{dt}\left({}^{I}\boldsymbol{\omega}^{S_1} + {}^{S_1}\boldsymbol{\omega}^{S_2}\right) = \frac{^{I}d\ ^{I}\boldsymbol{\omega}^{S_1}}{dt} + \frac{^{I}d\ ^{S_1}\boldsymbol{\omega}^{S_2}}{dt}$$

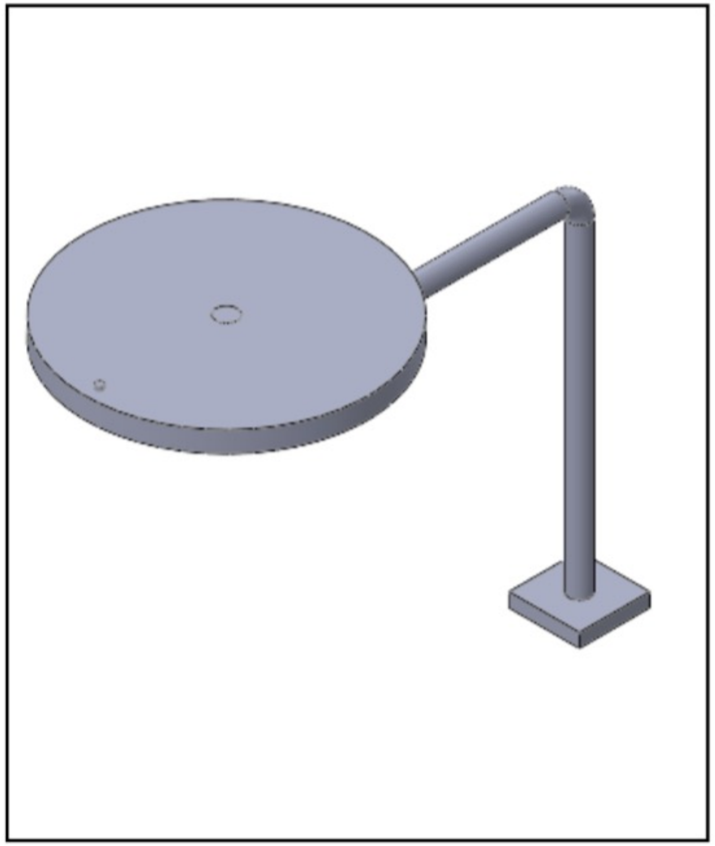
$$= \frac{^{I}d\ ^{I}\boldsymbol{\omega}^{S_1}}{dt} + \frac{^{S_1}d\ ^{S_1}\boldsymbol{\omega}^{S_2}}{dt} + {}^{I}\boldsymbol{\omega}^{S_1} \times {}^{S_1}\boldsymbol{\omega}^{S_2}$$

$$^{I}\boldsymbol{\alpha}^{S_2} = {}^{I}\boldsymbol{\alpha}^{S_1} + {}^{S_1}\boldsymbol{\alpha}^{S_2} + {}^{I}\boldsymbol{\omega}^{S_1} \times {}^{S_1}\boldsymbol{\omega}^{S_2}$$

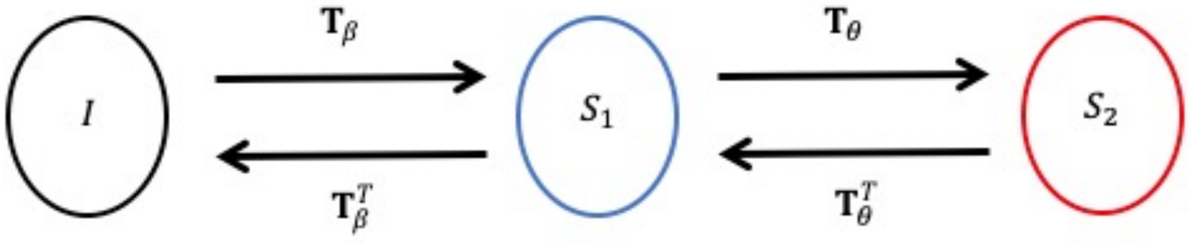
$$^{I}\boldsymbol{\alpha}^{S_2} \neq {}^{I}\boldsymbol{\alpha}^{S_1} + {}^{S_1}\boldsymbol{\alpha}^{S_2}$$

Velocidade e Aceleração Angulares

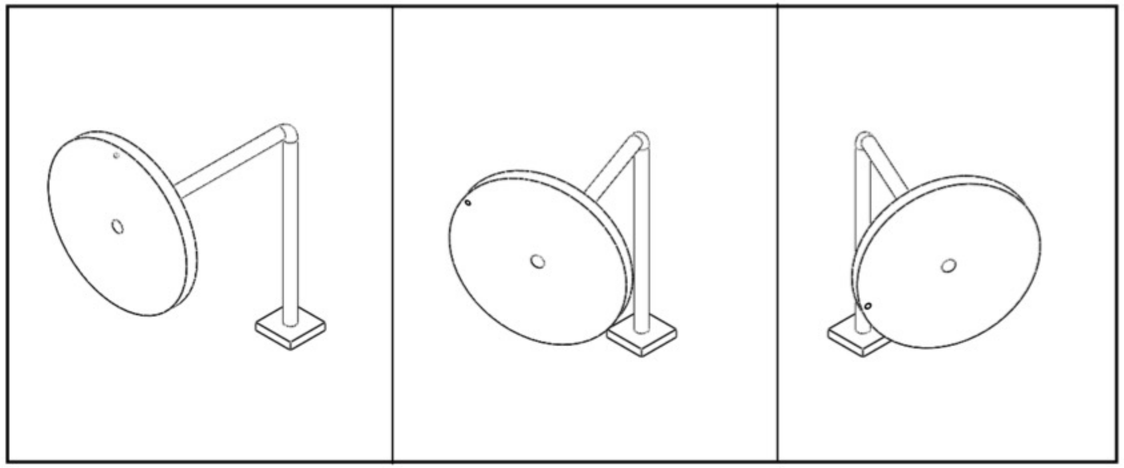


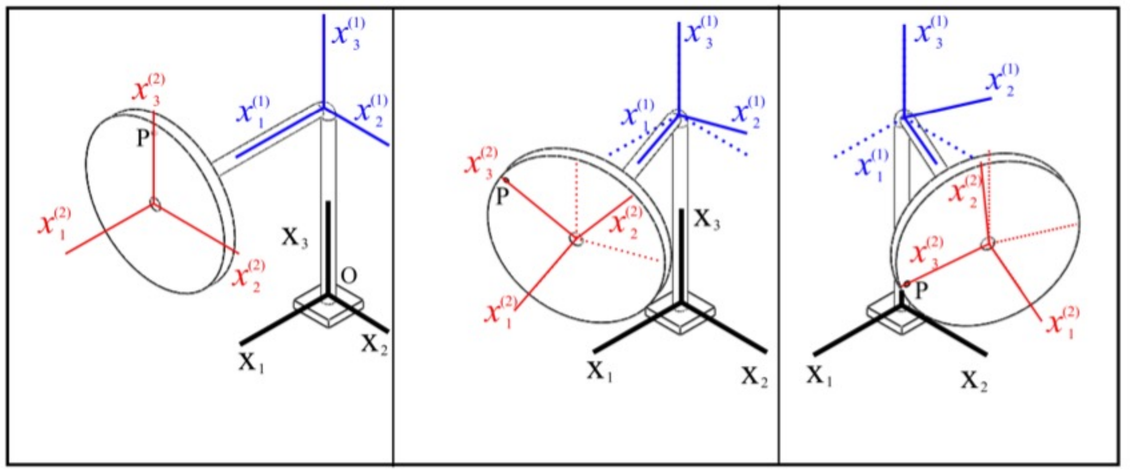


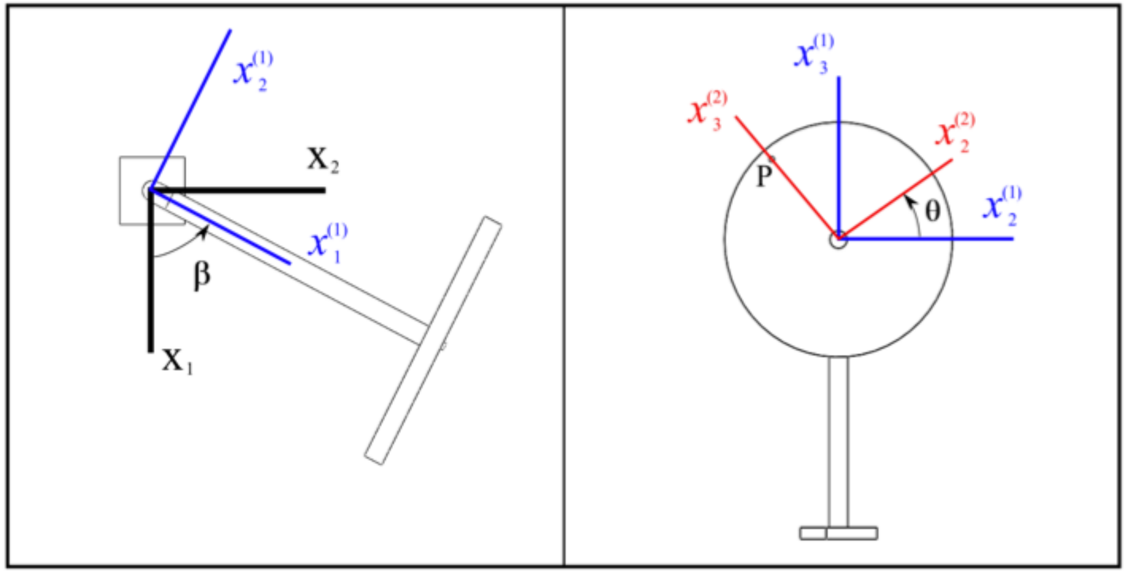
- Inercial, I (X_i)
- Móvel 1, solidário à haste em L, S_1 ($x_i^{(1)}$)
- Móvel 2, solidário ao disco, S_2 ($x_i^{(2)}$)



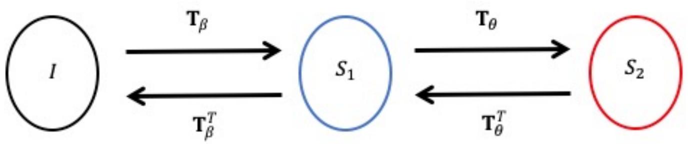
Velocidade e Aceleração Angulares





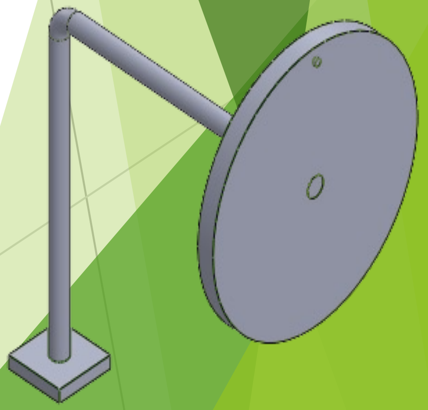


Tipo 3-1

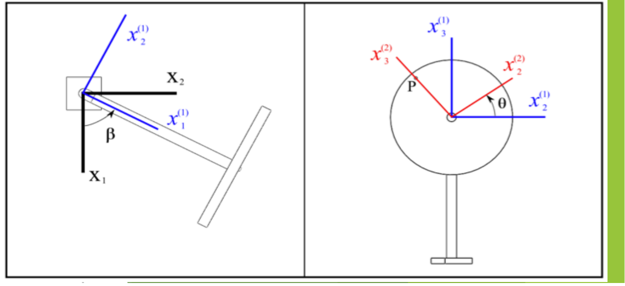


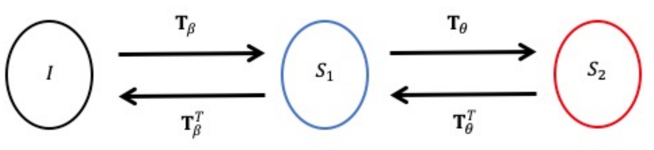
Sistema S_1: $\mathbf{T}_\beta = \begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Sistema S_2: $\mathbf{T}_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix}$



Velocidade Angular





$$ {}^{I}_{I}\boldsymbol{\omega}^{S_2} = {}^{I}_{I}\boldsymbol{\omega}^{S_1} + {}^{S_1}_{I}\boldsymbol{\omega}^{S_2} $$

$$ {}^{I}_{I}\boldsymbol{\omega}^{S_1} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\beta} \end{Bmatrix} \qquad {}^{S_1}_{S_1}\boldsymbol{\omega}^{S_2} = \begin{Bmatrix} \dot{\theta} \\ 0 \\ 0 \end{Bmatrix} $$

- Base s_1

$$ {}^{I}_{S_1}\boldsymbol{\omega}^{S_2} = \mathbf{T}_\beta \, {}^{I}_{I}\boldsymbol{\omega}^{S_1} + {}^{S_1}_{S_1}\boldsymbol{\omega}^{S_2} $$

$$ {}^{I}_{S_1}\boldsymbol{\omega}^{S_2} = \begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\beta} \end{Bmatrix} + \begin{Bmatrix} \dot{\theta} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \dot{\theta} \\ 0 \\ \dot{\beta} \end{Bmatrix} $$

$$ {}^{I}_{S_1}\boldsymbol{\omega}^{S_2} = \dot{\theta}\mathbf{e}_1^{(1)} + \dot{\beta}\,\mathbf{e}_3^{(1)} $$

- Base s_2

$$ {}^{I}_{S_2}\boldsymbol{\omega}^{S_2} = \mathbf{T}_\theta(\mathbf{T}_\beta \, {}^{I}_{I}\boldsymbol{\omega}^{S_1}) + \mathbf{T}_\theta \, {}^{S_1}_{S_1}\boldsymbol{\omega}^{S_2} $$

$$ {}^{I}_{S_2}\boldsymbol{\omega}^{S_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix} \begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\beta} \end{Bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \dot{\theta} \\ \dot{\beta}\,s\theta \\ \dot{\beta}\,c\theta \end{Bmatrix} $$

$$ {}^{I}_{S_2}\boldsymbol{\omega}^{S_2} = \dot{\theta}\mathbf{e}_1^{(2)} + \dot{\beta}\,s\theta\,\mathbf{e}_2^{(2)} + \dot{\beta}\,c\theta\,\mathbf{e}_3^{(2)} $$

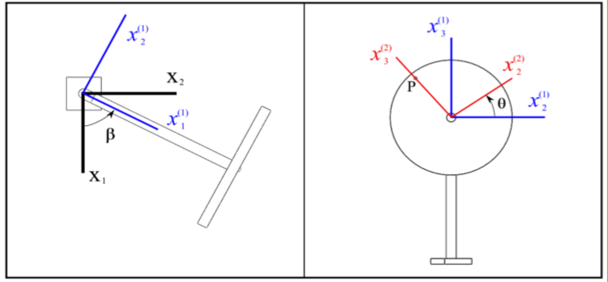
- Base I

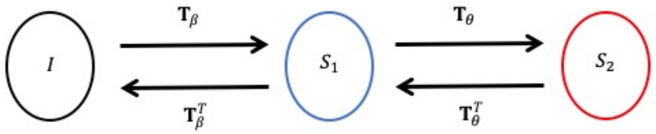
$$ {}^{I}_{I}\boldsymbol{\omega}^{S_2} = {}^{I}_{I}\boldsymbol{\omega}^{S_1} + \mathbf{T}_\beta^T \, {}^{S_1}_{S_1}\boldsymbol{\omega}^{S_2} $$

$$ {}^{I}_{I}\boldsymbol{\omega}^{S_2} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\beta} \end{Bmatrix} + \begin{bmatrix} c\beta & -s\beta & 0 \\ s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \dot{\theta}\,c\beta \\ \dot{\theta}\,s\beta \\ \dot{\beta} \end{Bmatrix} $$

$$ {}^{I}_{I}\boldsymbol{\omega}^{S_2} = \dot{\theta}\,c\beta\,\mathbf{E}_1 + \dot{\theta}\,s\beta\,\mathbf{E}_2 + \dot{\beta}\,\mathbf{E}_3 $$

Aceleração Angular





$$ {}_I^I\boldsymbol{\alpha}^{S_2} = {}_I^I\boldsymbol{\alpha}^{S_1} + {}_I^{S_1}\boldsymbol{\alpha}^{S_2} + {}_I^I\boldsymbol{\omega}^{S_1} \times {}_I^{S_1}\boldsymbol{\omega}^{S_2} $$

$$ {}_I^I\boldsymbol{\alpha}^{S_1} = \begin{Bmatrix} 0 \\ 0 \\ \ddot{\beta} \end{Bmatrix} \qquad {}_{S_1}^{S_1}\boldsymbol{\alpha}^{S_2} = \begin{Bmatrix} \ddot{\theta} \\ 0 \\ 0 \end{Bmatrix} $$

- Base S_1

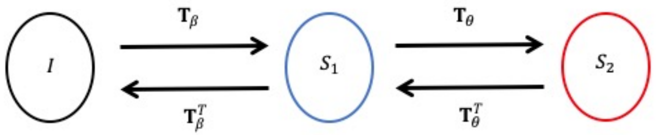
$$ {}_{S_1}^I\boldsymbol{\alpha}^{S_2} = \mathbf{T}_\beta \, {}_I^I\boldsymbol{\alpha}^{S_1} + {}_{S_1}^{S_1}\boldsymbol{\alpha}^{S_2} + {}_{S_1}^I\boldsymbol{\omega}^{S_1} \times {}_{S_1}^{S_1}\boldsymbol{\omega}^{S_2} $$

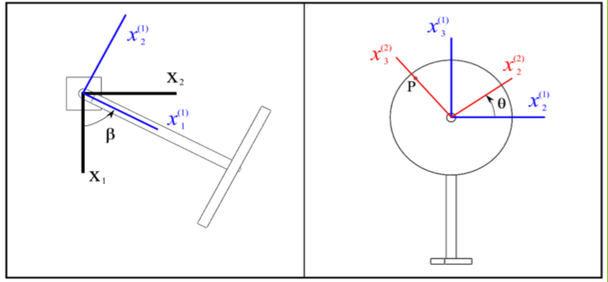
$$ {}_{S_1}^I\boldsymbol{\alpha}^{S_2} = \begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \ddot{\beta} \end{Bmatrix} + \begin{Bmatrix} \ddot{\theta} \\ 0 \\ 0 \end{Bmatrix} + \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \dot{\beta} \\ \dot{\theta} & 0 & 0 \end{vmatrix} = \begin{Bmatrix} \ddot{\theta} \\ \dot{\beta}\dot{\theta} \\ \ddot{\beta} \end{Bmatrix} $$

$$ {}_{S_1}^I\boldsymbol{\alpha}^{S_2} = \begin{Bmatrix} 0 \\ \dot{\beta}\dot{\theta} \\ 0 \end{Bmatrix} $$

Existe aceleração angular, mesmo quando $\ddot{\theta} = \ddot{\beta} = 0$!!

Aceleração Angular





- Base S_2

$$ {}_{S_2}^{I}\boldsymbol{\alpha}^{S_2} = \mathbf{T}_\theta \left(\mathbf{T}_\beta \; {}_{I}^{I}\boldsymbol{\alpha}^{S_1}\right) + \mathbf{T}_\theta \, {}_{S_1}^{S_1}\boldsymbol{\alpha}^{S_2} + \; {}_{S_2}^{I}\boldsymbol{\omega}^{S_1} \times {}_{S_2}^{S_1}\boldsymbol{\omega}^{S_2} $$

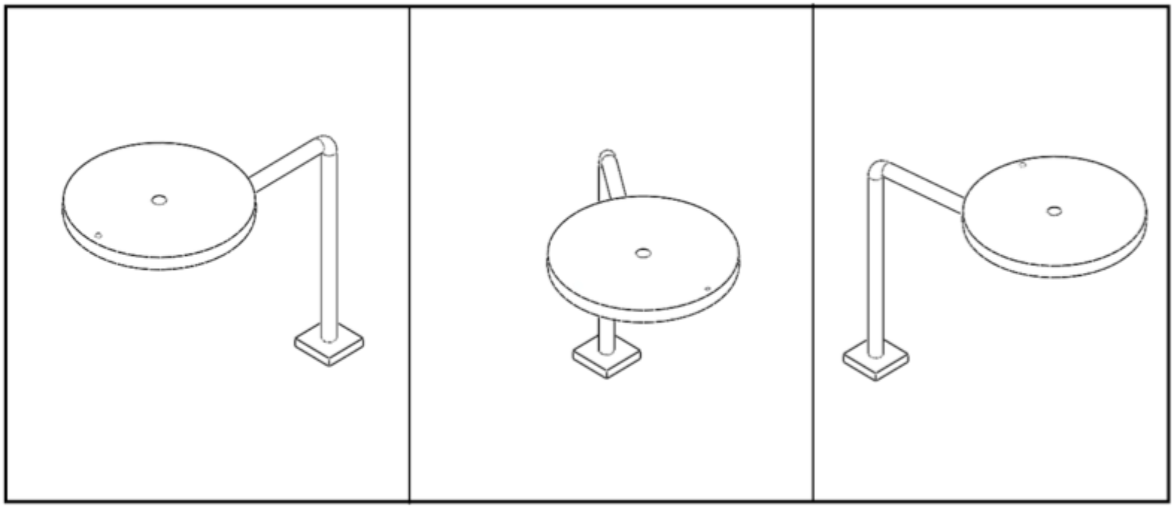
$$ {}_{S_2}^{I}\boldsymbol{\alpha}^{S_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix} \begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \ddot{\beta} \end{Bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ 0 \\ 0 \end{Bmatrix} + \begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ 0 & \dot{\beta}\, s\theta & \dot{\beta}\, c\theta \\ \dot{\theta} & 0 & 0 \end{vmatrix} = \begin{Bmatrix} \ddot{\theta} \\ \ddot{\beta}\, s\theta + \dot{\beta}\dot{\theta}\, c\theta \\ \ddot{\beta}\, c\theta - \dot{\beta}\dot{\theta}\, s\theta \end{Bmatrix} $$

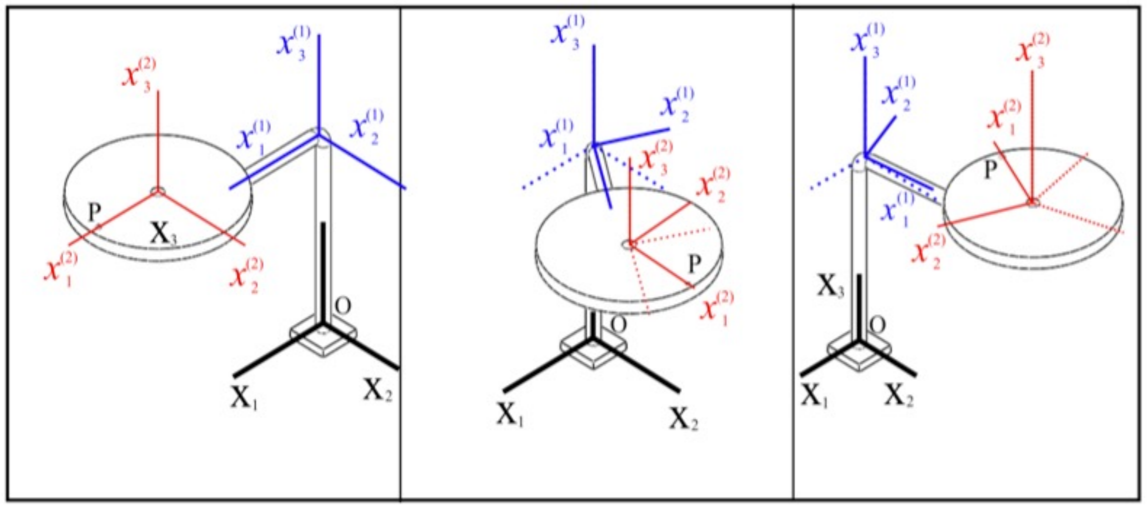
- Base I

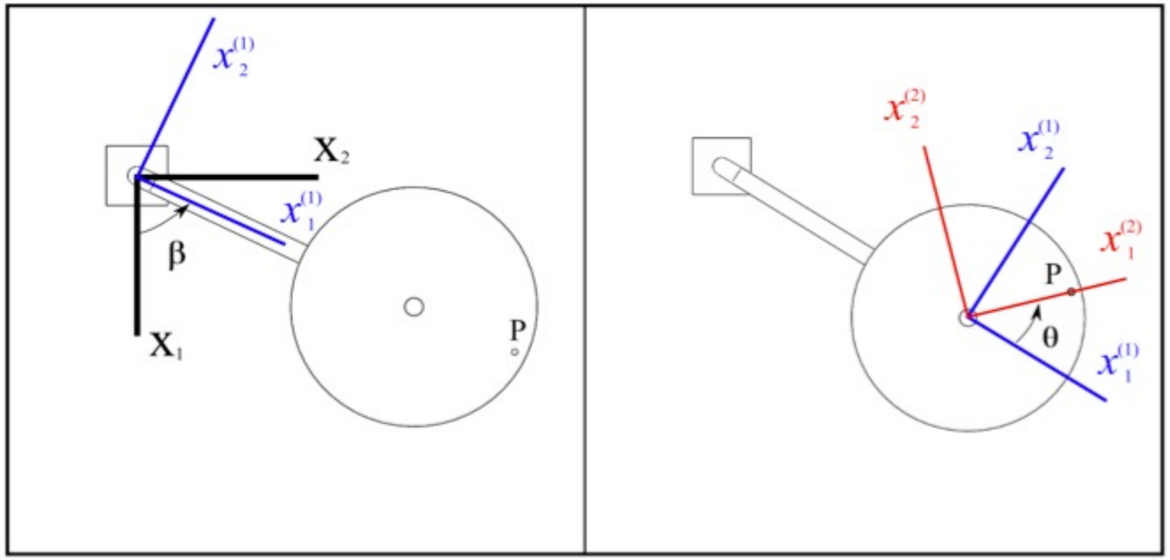
$$ {}_{I}^{I}\boldsymbol{\alpha}^{S_2} = \mathbf{T}_\beta^T \mathbf{T}_\theta^T \; {}_{S_2}^{I}\boldsymbol{\alpha}^{S_2} $$

$$ {}_{I}^{I}\boldsymbol{\alpha}^{S_2} = \begin{bmatrix} c\beta & -s\beta & 0 \\ s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{\beta}\, s\theta + \dot{\beta}\dot{\theta}\, c\theta \\ \ddot{\beta}\, c\theta - \dot{\beta}\dot{\theta}\, s\theta \end{Bmatrix} = \begin{Bmatrix} \ddot{\theta}\, c\beta - \dot{\beta}\dot{\theta}\, s\beta \\ \ddot{\theta}\, s\beta + \dot{\beta}\dot{\theta}\, s\beta \\ \ddot{\beta} \end{Bmatrix} $$

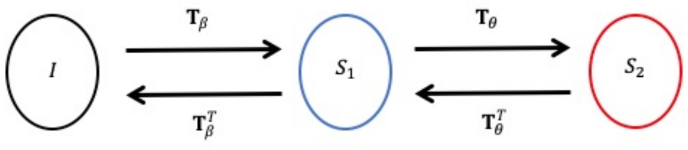
Velocidade e Aceleração Angulares





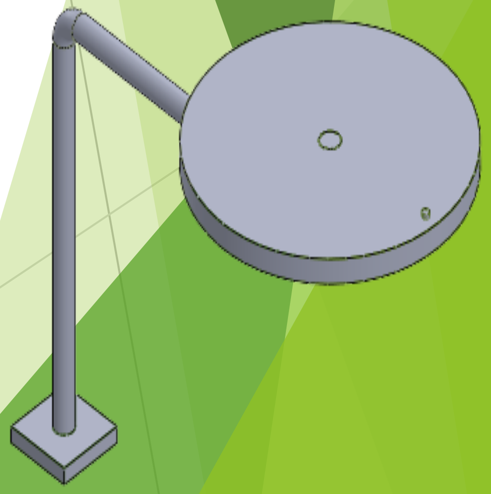


Tipo 3-3

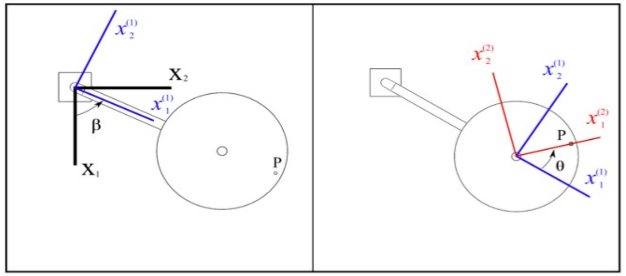


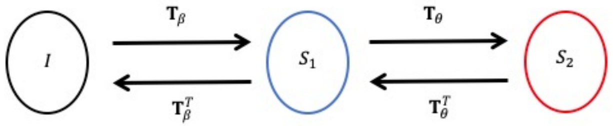
Sistema S_1: $\mathbf{T}_\beta = \begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Sistema S_2: $\mathbf{T}_\theta = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$



Velocidade Angular





$$ {}_{I}^{I}\boldsymbol{\omega}^{S_2} = {}_{I}^{I}\boldsymbol{\omega}^{S_1} + {}_{I}^{S_1}\boldsymbol{\omega}^{S_2} $$

$$ {}_{I}^{I}\boldsymbol{\omega}^{S_1} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\beta} \end{Bmatrix} \qquad {}_{S_1}^{S_1}\boldsymbol{\omega}^{S_2} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} $$

- Base S_1

$$ {}_{S_1}^{I}\boldsymbol{\omega}^{S_2} = \mathbf{T}_\beta \, {}_{I}^{I}\boldsymbol{\omega}^{S_1} + {}_{S_1}^{S_1}\boldsymbol{\omega}^{S_2} $$

$$ {}_{S_1}^{I}\boldsymbol{\omega}^{S_2} = \begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\beta} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\beta} + \dot{\theta} \end{Bmatrix} $$

- Base S_2

$$ {}_{S_2}^{I}\boldsymbol{\omega}^{S_2} = \mathbf{T}_\theta (\mathbf{T}_\beta \, {}_{I}^{I}\boldsymbol{\omega}^{S_1}) + \mathbf{T}_\theta \, {}_{S_1}^{S_1}\boldsymbol{\omega}^{S_2} $$

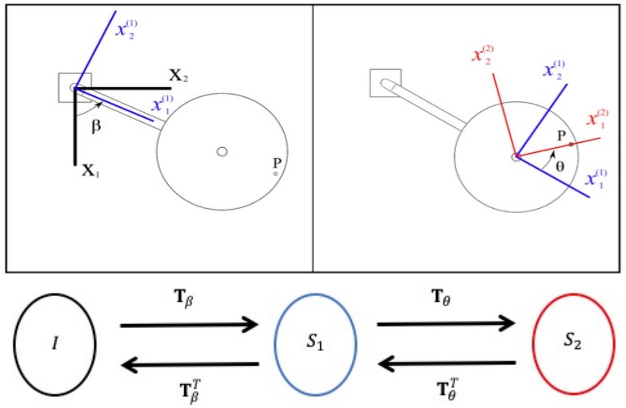
$$ {}_{S_2}^{I}\boldsymbol{\omega}^{S_2} = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\beta} \end{Bmatrix} + \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\beta} + \dot{\theta} \end{Bmatrix} $$

- Base I

$$ {}_{I}^{I}\boldsymbol{\omega}^{S_2} = {}_{I}^{I}\boldsymbol{\omega}^{S_1} + \mathbf{T}_\beta^T \, {}_{S_1}^{S_1}\boldsymbol{\omega}^{S_2} $$

$$ {}_{I}^{I}\boldsymbol{\omega}^{S_2} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\beta} \end{Bmatrix} + \begin{bmatrix} c\beta & -s\beta & 0 \\ s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\beta} + \dot{\theta} \end{Bmatrix} $$

Aceleração Angular



$${}_{I}^{I}\boldsymbol{\alpha}^{S_2} = {}_{I}^{I}\boldsymbol{\alpha}^{S_1} + {}_{I}^{S_1}\boldsymbol{\alpha}^{S_2} + {}_{I}^{I}\boldsymbol{\omega}^{S_1} \times {}_{I}^{S_1}\boldsymbol{\omega}^{S_2}$$

$${}_{I}^{I}\boldsymbol{\omega}^{S_2} \; // \; {}_{I}^{S_1}\boldsymbol{\omega}^{S_2} \quad 0$$

$${}_{I}^{I}\boldsymbol{\alpha}^{S_1} = \begin{Bmatrix} 0 \\ 0 \\ \ddot{\beta} \end{Bmatrix} \qquad {}_{S_1}^{S_1}\boldsymbol{\alpha}^{S_2} = \begin{Bmatrix} 0 \\ 0 \\ \ddot{\theta} \end{Bmatrix}$$

- Base S_1

$${}_{S_1}^{I}\boldsymbol{\alpha}^{S_2} = \mathbf{T}_\beta \, {}_{I}^{I}\boldsymbol{\alpha}^{S_1} + {}_{S_1}^{S_1}\boldsymbol{\alpha}^{S_2}$$

$${}_{S_1}^{I}\boldsymbol{\alpha}^{S_2} = \begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \ddot{\beta} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \ddot{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \ddot{\beta} + \ddot{\theta} \end{Bmatrix}$$

- Base S_2

$${}_{S_2}^{I}\boldsymbol{\alpha}^{S_2} = \mathbf{T}_\theta \left(\mathbf{T}_\beta \, {}_{I}^{I}\boldsymbol{\alpha}^{S_1} \right) + \mathbf{T}_\theta \, {}_{S_1}^{S_1}\boldsymbol{\alpha}^{S_2} = \begin{Bmatrix} 0 \\ 0 \\ \ddot{\beta} + \ddot{\theta} \end{Bmatrix}$$

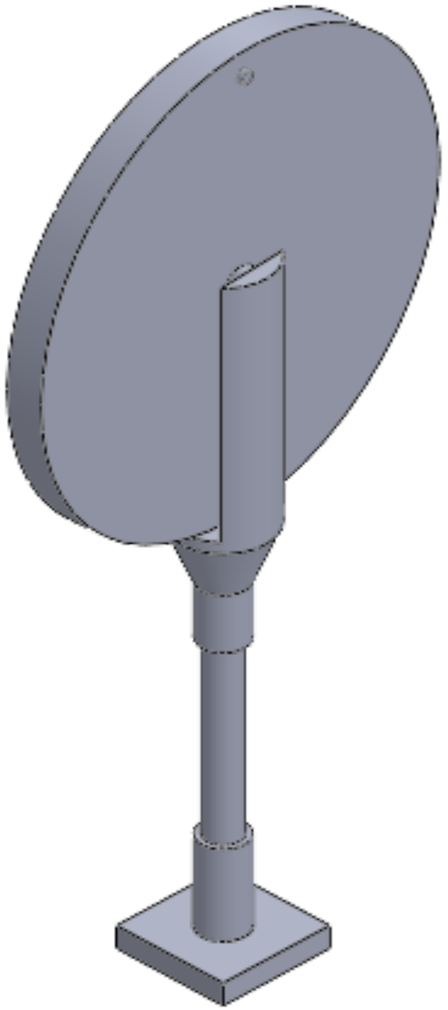
- Base I

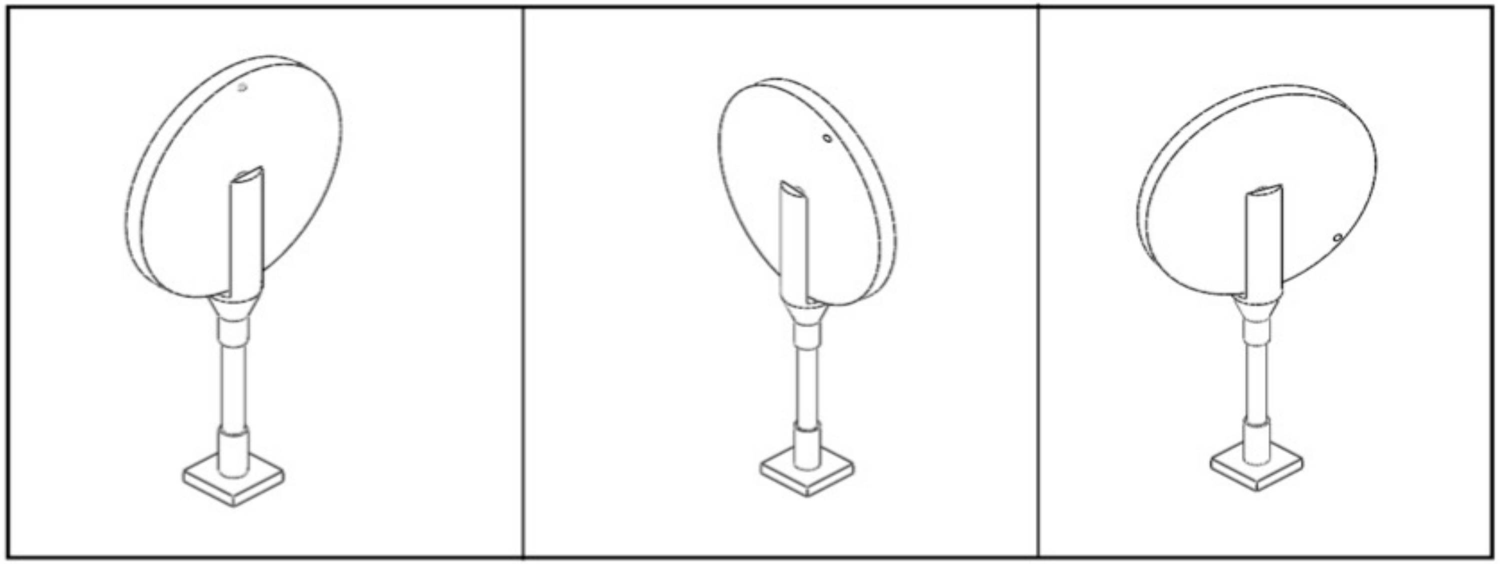
$${}_{I}^{I}\boldsymbol{\alpha}^{S_2} = \begin{bmatrix} c\beta & -s\beta & 0 \\ s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \ddot{\beta} + \ddot{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \ddot{\beta} + \ddot{\theta} \end{Bmatrix}$$

Não existe aceleração angular quando $\ddot{\theta} = \ddot{\beta} = 0$!

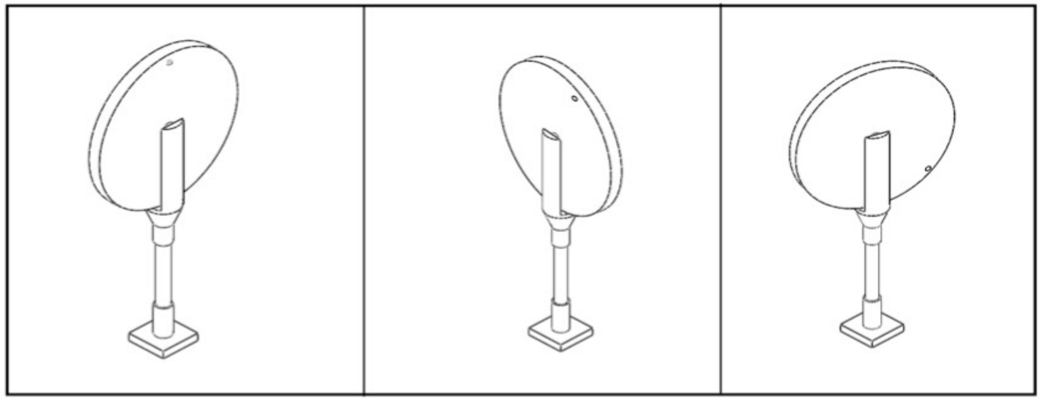
As aceleração angulares são sempre perpendiculares ao plano!

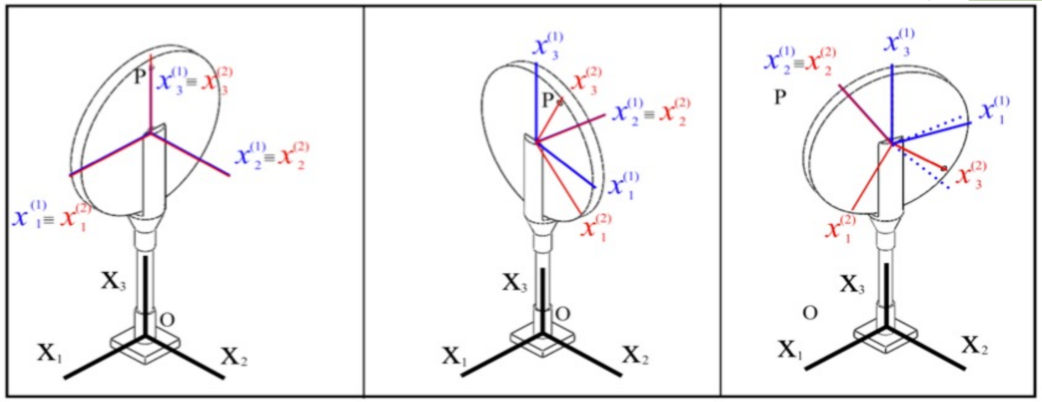
Sistema Tipo 3-2

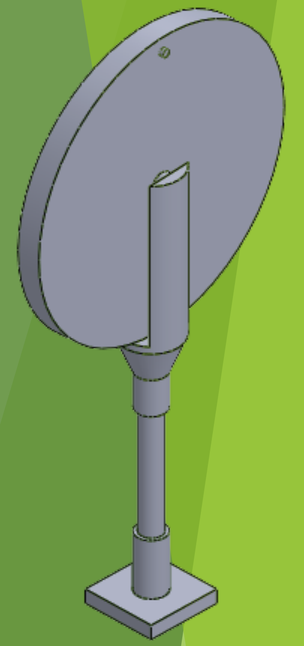


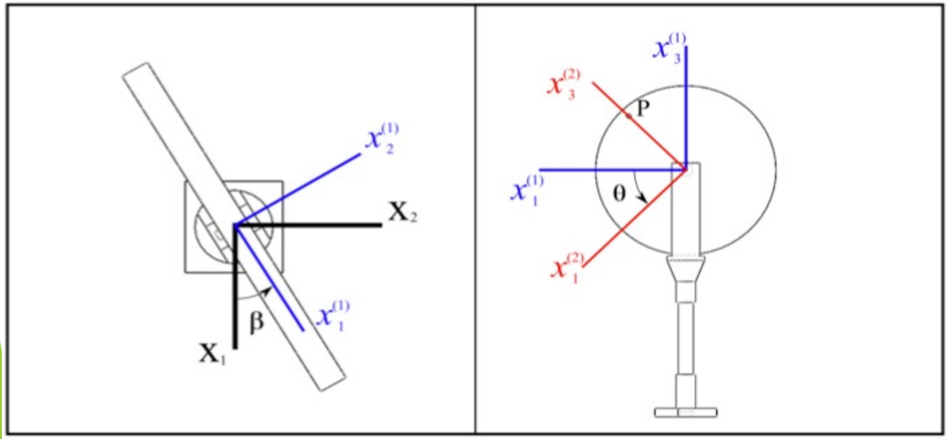


Sistema Tipo 3-2







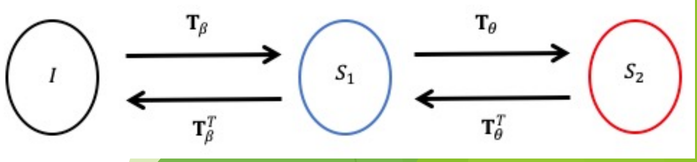


- Inercial, I (X_i)
- Móvel 1, solidário à haste, S_1 $(x_i^{(1)})$
- Móvel 2, solidário ao disco, S_2 $(x_i^{(2)})$

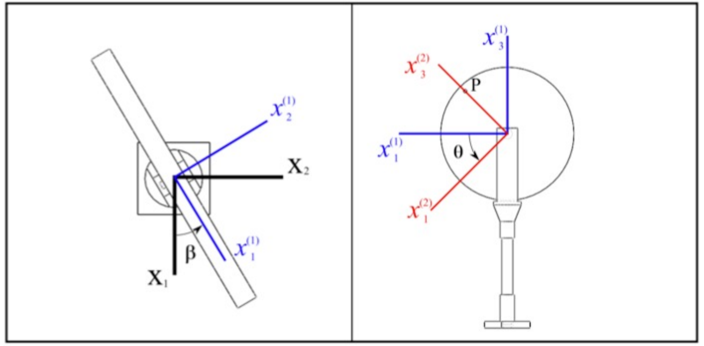
Sistema S_1: $\mathbf{T}_\beta = \begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

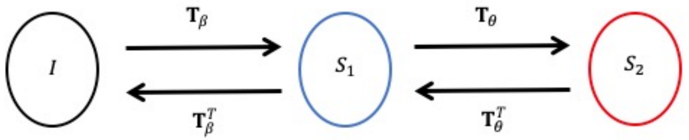
Sistema S_2: $\mathbf{T}_\theta = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}$

Tipo 3-2



Sistema Tipo 3-2: Velocidade e Aceleração Angulares





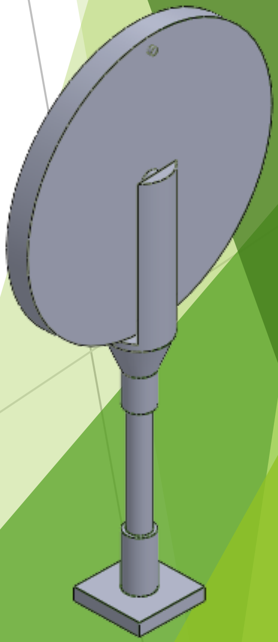
$$ {}^{I}\boldsymbol{\omega}^{S_2} = {}^{I}\boldsymbol{\omega}^{S_1} + {}^{S_1}\boldsymbol{\omega}^{S_2} $$

$$ {}^{I}_{S_1}\boldsymbol{\omega}^{S_1} = \mathbf{T}_\beta \, {}^{I}_{I}\boldsymbol{\omega}^{S_1} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\beta} \end{Bmatrix} $$

$$ {}^{S_1}_{S_1}\boldsymbol{\omega}^{S_2} = \begin{Bmatrix} 0 \\ \dot{\theta} \\ 0 \end{Bmatrix} $$

$$ {}^{I}_{S_1}\boldsymbol{\omega}^{S_2} = {}^{I}_{S_1}\boldsymbol{\omega}^{S_1} + {}^{S_1}_{S_1}\boldsymbol{\omega}^{S_2} = \begin{Bmatrix} 0 \\ \dot{\theta} \\ \dot{\beta} \end{Bmatrix} $$

$$ {}^{I}_{S_1}\boldsymbol{\alpha}^{S_2} = {}^{I}_{S_1}\boldsymbol{\alpha}^{S_1} + {}^{S_1}_{S_1}\boldsymbol{\alpha}^{S_2} + {}^{I}_{S_1}\boldsymbol{\omega}^{S_1} \times {}^{S_1}_{S_1}\boldsymbol{\omega}^{S_2} = \begin{Bmatrix} 0 \\ 0 \\ \ddot{\beta} \end{Bmatrix} + \begin{Bmatrix} 0 \\ \ddot{\theta} \\ 0 \end{Bmatrix} + \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \dot{\beta} \\ 0 & \dot{\theta} & 0 \end{vmatrix} = \begin{Bmatrix} -\dot{\beta}\dot{\theta} \\ \ddot{\theta} \\ \ddot{\beta} \end{Bmatrix} $$



Sistema Tipo 3-2: Velocidade

- Alternativa usando o referencial S_1

$${}^{I}_{S_1}\mathbf{v}^P = {}_{S_1}\mathbf{v}^{O_1} + {}^{S_1}_{S_1}\mathbf{v}^P + {}^{I}_{S_1}\boldsymbol{\omega}^{S_1} \times {}^{S_1}_{S_1}\mathbf{p}^P$$

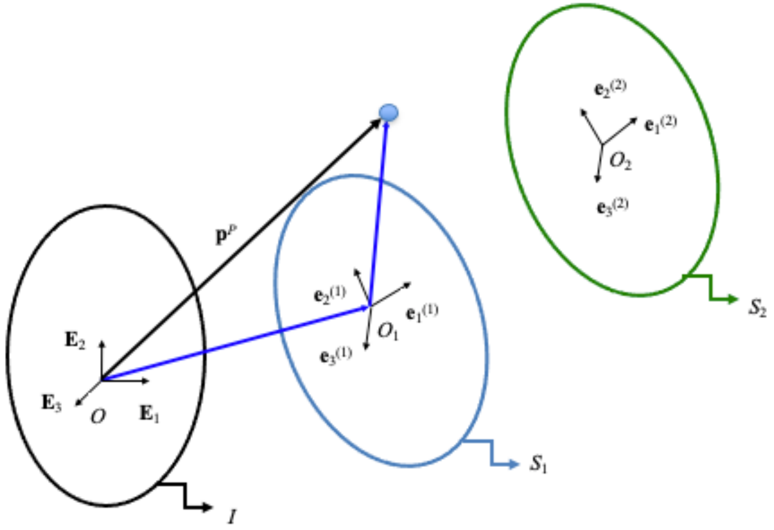
$${}^{S_1}_{S_1}\mathbf{v}^P = {}^{S_1}_{S_1}\mathbf{v}^{O_2} + {}^{S_2}_{S_1}\mathbf{v}^P + {}^{S_1}_{S_1}\boldsymbol{\omega}^{S_2} \times {}^{S_2}_{S_1}\mathbf{p}^P$$

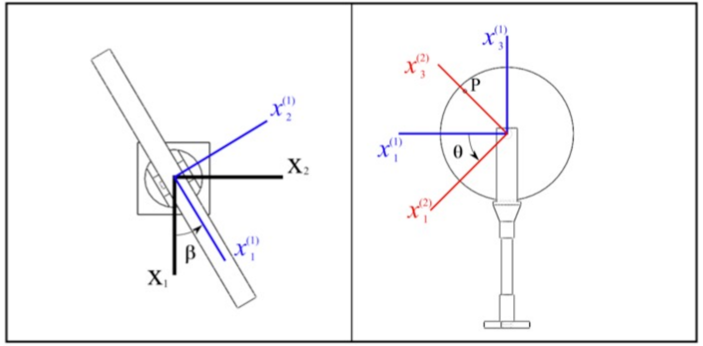
0 0

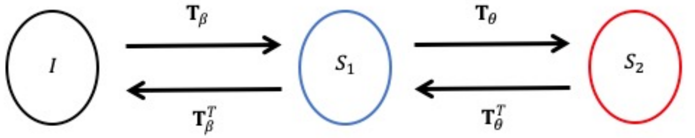
$${}^{S_1}_{S_1}\boldsymbol{\omega}^{S_2} = \begin{Bmatrix} 0 \\ \dot{\theta} \\ 0 \end{Bmatrix}$$

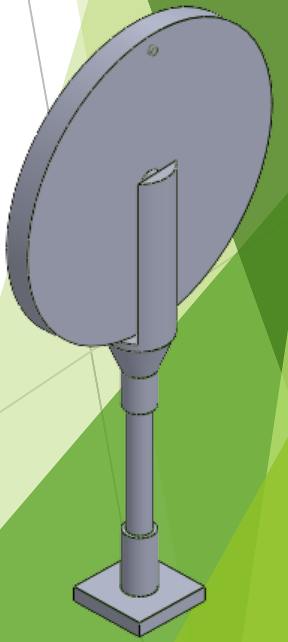
$${}^{S_2}_{S_1}\mathbf{p}^P = \mathbf{T}_\theta^T \, {}^{S_2}_{S_2}\mathbf{p}^P = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ R \end{Bmatrix} = \begin{Bmatrix} R\, s\theta \\ 0 \\ R\, c\theta \end{Bmatrix}$$

$${}^{S_1}_{S_1}\mathbf{v}^P = {}^{S_1}_{S_1}\boldsymbol{\omega}^{S_2} \times {}^{S_2}_{S_1}\mathbf{p}^P = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & \dot{\theta} & 0 \\ R\, s\theta & 0 & R\, c\theta \end{vmatrix} = \begin{Bmatrix} R\dot{\theta}\, c\theta \\ 0 \\ -R\dot{\theta}\, s\theta \end{Bmatrix}$$









Sistema Tipo 3-2: Velocidade

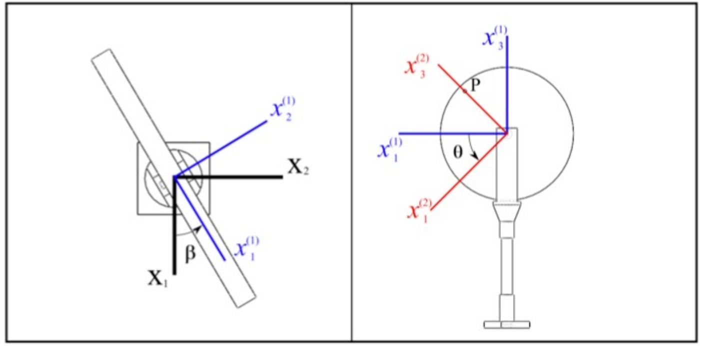
$${}^{I}_{S_1}\mathbf{v}^{P} = {}_{S_1}\mathbf{v}^{O_1} + {}^{S_1}_{S_1}\mathbf{v}^{P} + {}^{I}_{S_1}\boldsymbol{\omega}^{S_1}\times{}^{S_1}_{S_1}\mathbf{p}^{P}$$

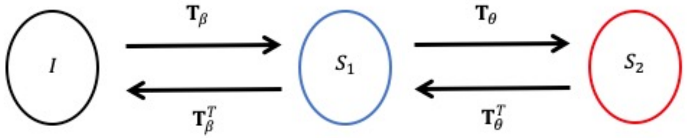
0

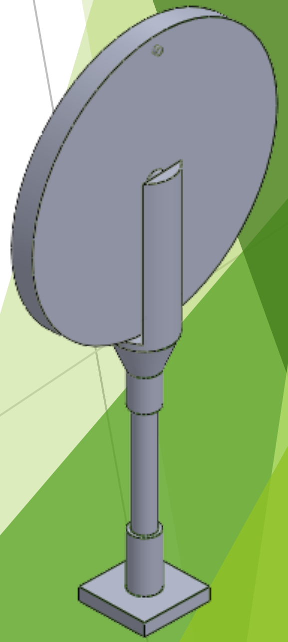
$${}^{I}_{S_1}\boldsymbol{\omega}^{S_1} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\beta} \end{Bmatrix}$$

$${}^{I}_{S_1}\boldsymbol{\omega}^{S_1}\times{}^{S_1}_{S_1}\mathbf{p}^{P} = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \dot{\beta} \\ R\,s\theta & 0 & R\,c\theta \end{vmatrix} = \begin{Bmatrix} 0 \\ R\dot{\beta}\,s\theta \\ 0 \end{Bmatrix}$$

$${}^{I}_{S_1}\mathbf{v}^{P} = \begin{Bmatrix} R\dot{\theta}\,c\theta \\ R\dot{\beta}\,s\theta \\ -R\dot{\theta}\,s\theta \end{Bmatrix}$$







Sistema Tipo 3-2: Velocidade

- Alternativa usando o referencial S_2

$${}^{I}_{S_1}\mathbf{v}^P = {}_{S_1}\mathbf{v}^{O_2} + {}^{S_2}_{S_1}\mathbf{v}^P + {}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_1}\mathbf{p}^P$$

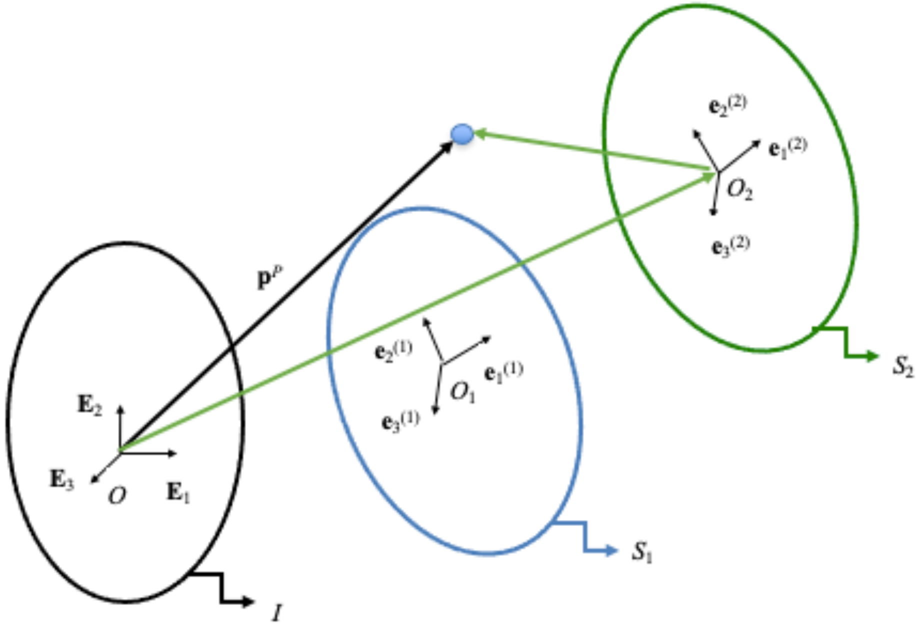
0 0

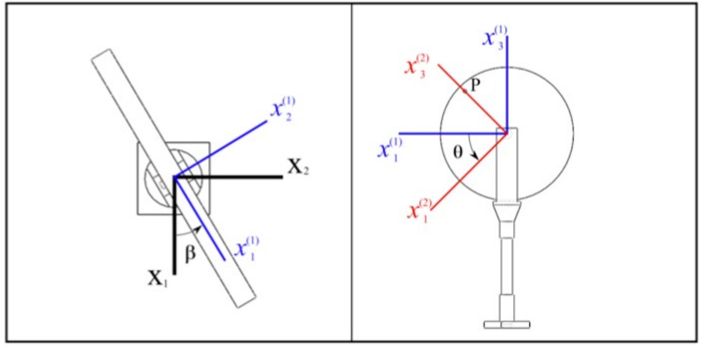
$${}^{I}\boldsymbol{\omega}^{S_2} = {}^{I}\boldsymbol{\omega}^{S_1} + {}^{S_1}\boldsymbol{\omega}^{S_2}$$

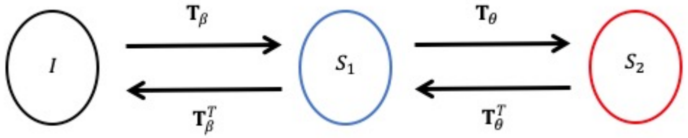
$${}^{I}_{S_1}\boldsymbol{\omega}^{S_1} = \mathbf{T}_\beta\,{}^{I}_{I}\boldsymbol{\omega}^{S_1} = \begin{Bmatrix}0\\0\\\dot{\beta}\end{Bmatrix}$$

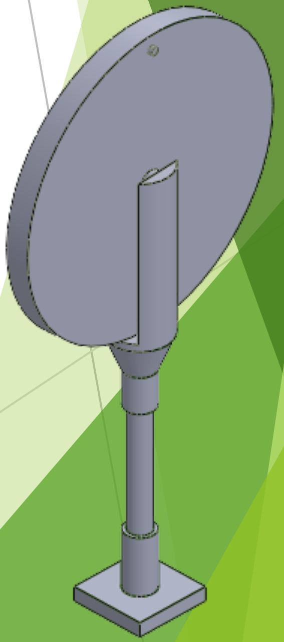
$${}^{S_1}_{I}\boldsymbol{\omega}^{S_2} = {}^{S_1}_{S_1}\boldsymbol{\omega}^{S_2} = \begin{Bmatrix}0\\\dot{\theta}\\0\end{Bmatrix}$$

$${}^{I}_{S_1}\mathbf{v}^P = {}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_1}\mathbf{p}^P = \begin{vmatrix}\mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)}\\ 0 & \dot{\theta} & \dot{\beta}\\ R\,s\theta & 0 & R\,c\theta\end{vmatrix} = \begin{Bmatrix}R\dot{\theta}\,c\theta\\ R\dot{\beta}\,s\theta\\ -R\dot{\theta}\,s\theta\end{Bmatrix}$$









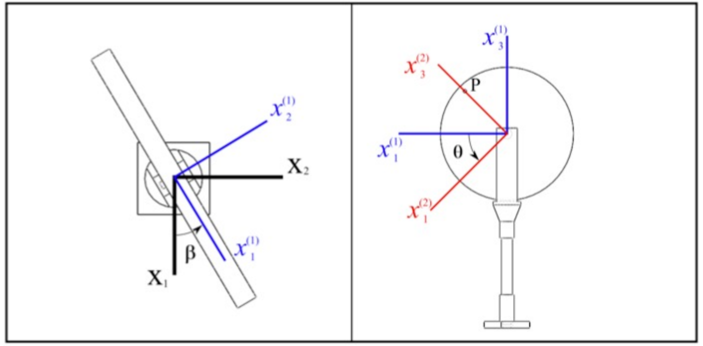
Sistema Tipo 3-2: Velocidade

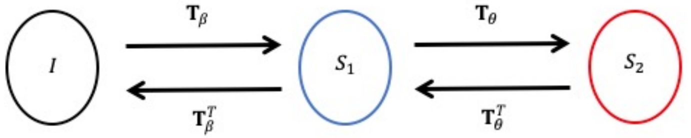
- Alternativa diferenciando o vetor posição

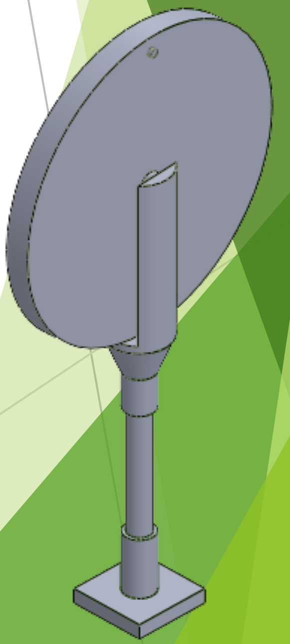
$$ {}^I_{S_1}\mathbf{v}^P = \frac{{}^I d\left({}^I_{S_1}\mathbf{p}^P\right)}{dt} = \frac{{}^{S_1} d\left({}^I_{S_1}\mathbf{p}^P\right)}{dt} + {}^I_{S_1}\boldsymbol{\omega}^{S_1} \times {}^I_{S_1}\mathbf{p}^P $$

$$ {}^I_{S_1}\mathbf{v}^P = \frac{{}^I d\left({}^I_{S_1}\mathbf{p}^P\right)}{dt} = \frac{{}^{S_1} d}{dt}\begin{Bmatrix} R\,s\theta \\ 0 \\ R\,c\theta \end{Bmatrix} + \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \dot{\beta} \\ R\,s\theta & 0 & R\,c\theta \end{vmatrix} $$

$$ {}^I_{S_1}\mathbf{v}^P = \begin{Bmatrix} R\dot{\theta}\,c\theta \\ R\dot{\beta}\,s\theta \\ -R\dot{\theta}\,s\theta \end{Bmatrix} $$

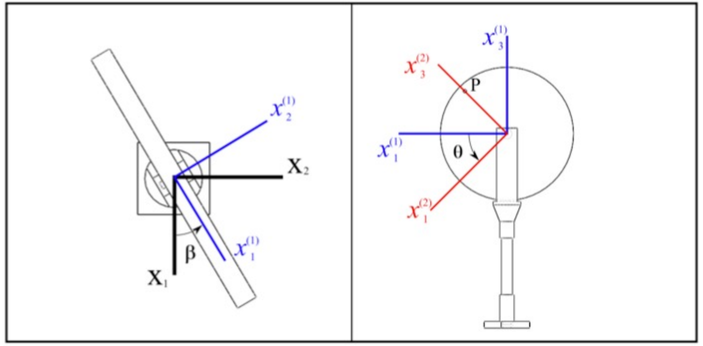


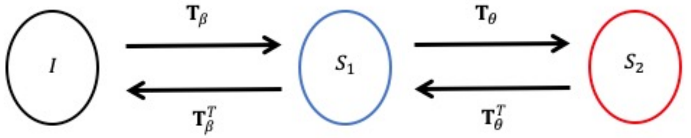




Sistema Tipo 3-2: Velocidade

- Avaliando a velocidade em diferentes bases

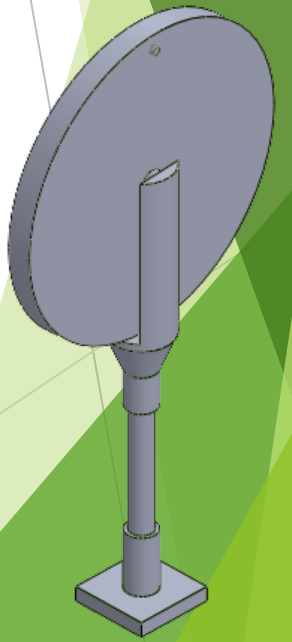




$$ {}^{I}_{S_1}\mathbf{v}^P = \begin{Bmatrix} R\dot{\theta}\, c\theta \\ R\dot{\beta}\, s\theta \\ -R\dot{\theta}\, s\theta \end{Bmatrix} $$

$$ {}^{I}_{I}\mathbf{v}^P = \mathbf{T}_\beta^T\, {}^{I}_{S_1}\mathbf{v}^P = \begin{bmatrix} c\beta & -s\beta & 0 \\ s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} R\dot{\theta}\, c\theta \\ R\dot{\beta}\, s\theta \\ -R\dot{\theta}\, s\theta \end{Bmatrix} = \begin{Bmatrix} R\dot{\theta}\, c\theta c\beta - R\dot{\beta}\, s\theta s\beta \\ R\dot{\theta}\, c\theta s\beta + R\dot{\beta}\, s\theta c\beta \\ -R\dot{\theta}\, s\theta \end{Bmatrix} $$

$$ {}^{I}_{S_2}\mathbf{v}^P = \mathbf{T}_\theta\, {}^{I}_{S_1}\mathbf{v}^P = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \begin{Bmatrix} R\dot{\theta}\, c\theta \\ R\dot{\beta}\, s\theta \\ -R\dot{\theta}\, s\theta \end{Bmatrix} = \begin{Bmatrix} R\dot{\theta}\, c^2\theta + R\dot{\theta}\, s^2\theta \\ R\dot{\beta}\, s\theta \\ R\dot{\theta}\, c\theta s\theta - R\dot{\theta}\, c\theta s\theta \end{Bmatrix} = \begin{Bmatrix} R\dot{\theta} \\ R\dot{\beta}\, s\theta \\ 0 \end{Bmatrix} $$



Sistema Tipo 3-2: Aceleração

- Usando o referencial S_1

$$ {}_{S_1}^{I}\mathbf{a}^{P} = {}_{S_1}^{I}\mathbf{a}^{O_1} + {}_{S_1}^{S_1}\mathbf{a}^{P} + {}_{S_1}^{I}\boldsymbol{\alpha}^{S_1}\times{}_{S_1}^{S_1}\mathbf{p}^{P} + {}_{S_1}^{I}\boldsymbol{\omega}^{S_1}\times\left({}_{S_1}^{I}\boldsymbol{\omega}^{S_1}\times{}_{S_1}^{S_1}\mathbf{p}^{P}\right) + 2\,{}_{S_1}^{I}\boldsymbol{\omega}^{S_1}\times{}_{S_1}^{S_1}\mathbf{v}^{P} $$

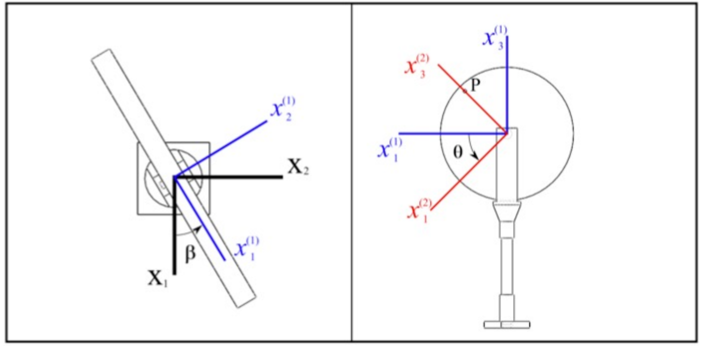
$$ {}_{S_1}^{S_1}\mathbf{a}^{P} = \underbrace{{}_{S_1}^{S_1}\mathbf{a}^{O_1}}_{0} + \underbrace{{}_{S_1}^{S_2}\mathbf{a}^{P}}_{0} + {}_{S_1}^{S_1}\boldsymbol{\alpha}^{S_2}\times{}_{S_1}^{S_1}\mathbf{p}^{P} + {}_{S_1}^{S_1}\boldsymbol{\omega}^{S_2}\times\left({}_{S_1}^{S_1}\boldsymbol{\omega}^{S_2}\times{}_{S_1}^{S_1}\mathbf{p}^{P}\right) + \underbrace{2\,{}_{S_1}^{S_1}\boldsymbol{\omega}^{S_2}\times{}_{S_1}^{S_2}\mathbf{v}^{P}}_{0} $$

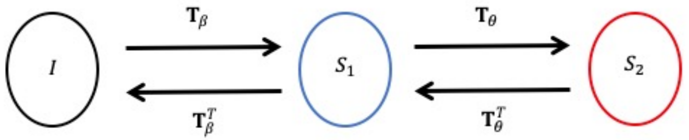
$$ {}_{S_1}^{S_1}\boldsymbol{\alpha}^{S_2} = {}_{S_1}^{I}\boldsymbol{\alpha}^{S_1} + {}_{S_1}^{S_1}\boldsymbol{\omega}^{S_2}\times{}_{S_1}^{S_1}\boldsymbol{\omega}^{S_2} = \begin{Bmatrix} 0 \\ \ddot{\theta} \\ 0 \end{Bmatrix} $$

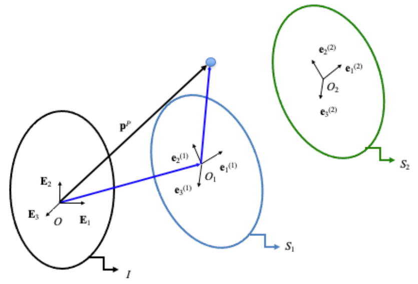
$$ {}_{S_1}^{S_1}\boldsymbol{\alpha}^{S_2}\times{}_{S_1}^{S_1}\mathbf{p}^{P} = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & \ddot{\theta} & 0 \\ R\,s\theta & 0 & R\,c\theta \end{vmatrix} = \begin{Bmatrix} R\ddot{\theta}\,c\theta \\ 0 \\ -R\ddot{\theta}\,s\theta \end{Bmatrix} $$

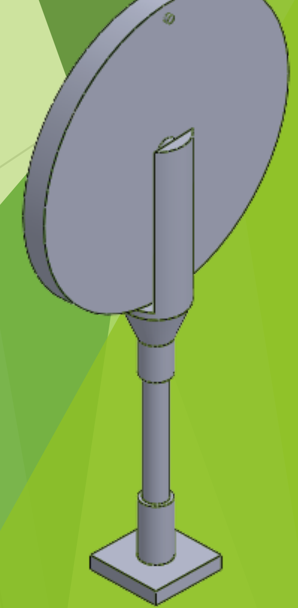
$$ {}_{S_1}^{S_1}\boldsymbol{\omega}^{S_2}\times\left({}_{S_1}^{S_1}\boldsymbol{\omega}^{S_2}\times{}_{S_1}^{S_1}\mathbf{p}^{P}\right) = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & \dot{\theta} & 0 \\ R\dot{\theta}\,c\theta & 0 & -R\dot{\theta}\,s\theta \end{vmatrix} = \begin{Bmatrix} -R\dot{\theta}^2\,s\theta \\ 0 \\ -R\dot{\theta}^2\,c\theta \end{Bmatrix} $$

$$ {}_{S_1}^{S_1}\mathbf{a}^{P} = \begin{Bmatrix} R\ddot{\theta}\,c\theta - R\dot{\theta}^2\,s\theta \\ 0 \\ -R\ddot{\theta}\,s\theta - R\dot{\theta}^2\,c\theta \end{Bmatrix} $$









Sistema Tipo 3-2: Aceleração

$$ {}^{I}_{S_1}\mathbf{a}^{P} = {}^{I}_{S_1}\mathbf{a}^{O_1} + {}^{S_1}_{S_1}\mathbf{a}^{P} + {}^{I}_{S_1}\boldsymbol{\alpha}^{S_1}\times{}^{S_1}_{S_1}\mathbf{p}^{P} + {}^{I}_{S_1}\boldsymbol{\omega}^{S_1}\times\left({}^{I}_{S_1}\boldsymbol{\omega}^{S_1}\times{}^{S_1}_{S_1}\mathbf{p}^{P}\right) + 2\,{}^{I}_{S_1}\boldsymbol{\omega}^{S_1}\times{}^{S_1}_{S_1}\mathbf{v}^{P} $$

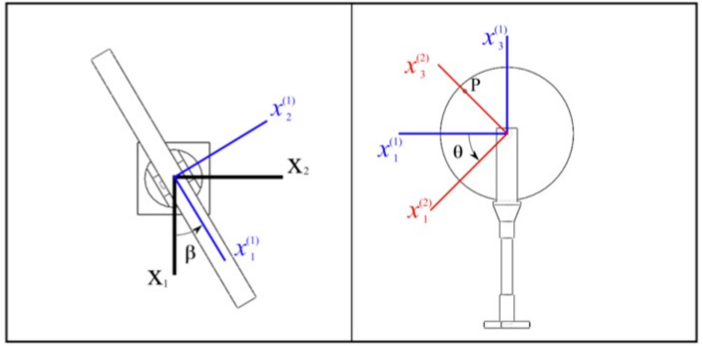
0

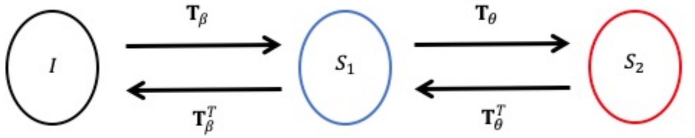
$$ {}^{I}_{S_1}\boldsymbol{\alpha}^{S_1}\times{}^{I}_{S_1}\mathbf{p}^{P} = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \ddot{\beta} \\ R\,s\theta & 0 & R\,c\theta \end{vmatrix} = \begin{Bmatrix} 0 \\ R\ddot{\beta}\,s\theta \\ 0 \end{Bmatrix} $$

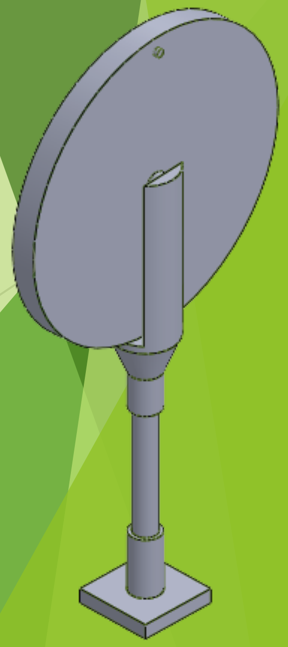
$$ {}^{I}_{S_1}\boldsymbol{\omega}^{S_1}\times\left({}^{I}_{S_1}\boldsymbol{\omega}^{S_1}\times{}^{I}_{S_1}\mathbf{p}^{P}\right) = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \dot{\beta} \\ 0 & R\dot{\beta}\,s\theta & 0 \end{vmatrix} = \begin{Bmatrix} -R\dot{\beta}^2\,s\theta \\ 0 \\ 0 \end{Bmatrix} $$

$$ 2\,{}^{I}_{S_1}\boldsymbol{\omega}^{S_1}\times{}^{S_1}_{S_1}\mathbf{v}^{P} = 2\begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & 0 & \dot{\beta} \\ R\dot{\theta}\,c\theta & 0 & -R\dot{\theta}\,s\theta \end{vmatrix} = \begin{Bmatrix} 0 \\ 2R\dot{\beta}\dot{\theta}\,c\theta \\ 0 \end{Bmatrix} $$

$$ {}^{I}_{S_1}\mathbf{a}^{P} = \begin{Bmatrix} R\ddot{\theta}\,c\theta - R(\dot{\theta}^2+\dot{\beta}^2)\,s\theta \\ R\ddot{\beta}\,s\theta + 2R\dot{\beta}\dot{\theta}\,c\theta \\ -R\ddot{\theta}\,s\theta - R\dot{\theta}^2\,c\theta \end{Bmatrix} $$







Sistema Tipo 3-2: Aceleração

- Usando o referencial S_2

$$ {}^{I}_{S_1}\mathbf{a}^P = {}^{I}_{S_1}\mathbf{a}^{O_2} + {}^{S_2}_{S_1}\mathbf{a}^P + {}^{I}_{S_1}\boldsymbol{\alpha}^{S_2}\times{}^{S_2}_{S_1}\mathbf{p}^P + {}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times\left({}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_1}\mathbf{p}^P\right) + 2\,{}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_1}\mathbf{v}^P $$

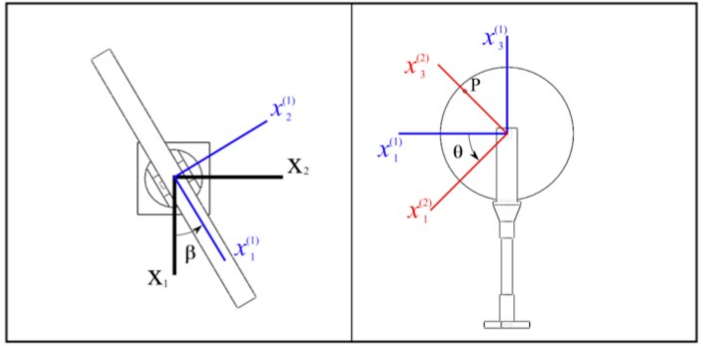
0 0 0

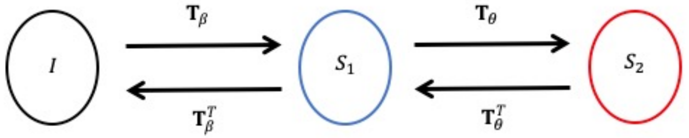
$$ {}^{I}_{S_1}\boldsymbol{\alpha}^{S_2}\times{}^{S_2}_{S_1}\mathbf{p}^P = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ -\dot{\beta}\dot{\theta} & \ddot{\theta} & \ddot{\beta} \\ R\,s\theta & 0 & R\,c\theta \end{vmatrix} = \begin{Bmatrix} R\ddot{\theta}\,c\theta \\ R\dot{\beta}\dot{\theta}\,c\theta + R\ddot{\beta}\,s\theta \\ -R\ddot{\theta}\,s\theta \end{Bmatrix} $$

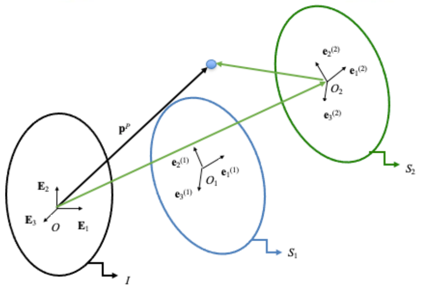
$$ {}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times\left({}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_1}\mathbf{p}^P\right) = {}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times\begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & \dot{\theta} & \dot{\beta} \\ R\,s\theta & 0 & R\,c\theta \end{vmatrix} = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & \dot{\theta} & \dot{\beta} \\ R\dot{\theta}\,c\theta & R\dot{\beta}\,s\theta & -R\dot{\theta}\,s\theta \end{vmatrix} $$

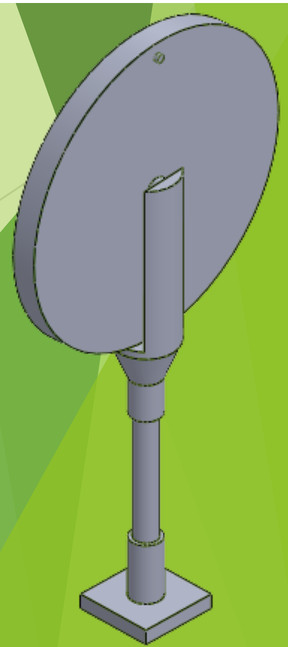
$$ = \begin{Bmatrix} -R\dot{\theta}^2\,s\theta - R\dot{\beta}^2\,s\theta \\ R\dot{\beta}\dot{\theta}\,c\theta \\ -R\dot{\theta}^2\,c\theta \end{Bmatrix} $$

$$ {}^{I}_{S_1}\mathbf{a}^P = \begin{Bmatrix} R\ddot{\theta}\,c\theta - R(\dot{\theta}^2 + \dot{\beta}^2)\,s\theta \\ R\ddot{\beta}\,s\theta + 2R\dot{\beta}\dot{\theta}\,c\theta \\ -R\ddot{\theta}\,s\theta - R\dot{\theta}^2\,c\theta \end{Bmatrix} $$





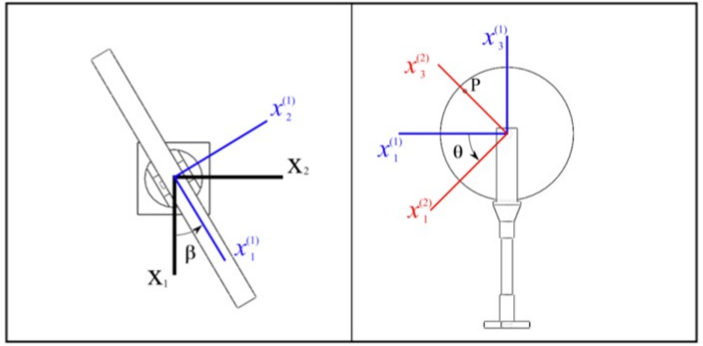


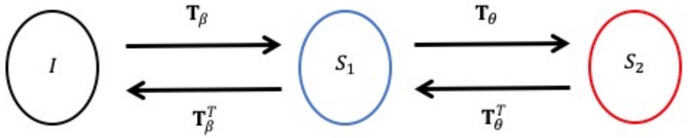


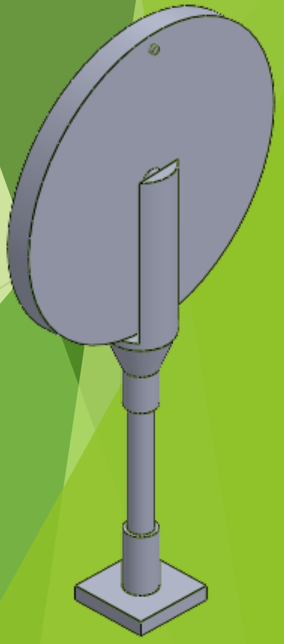
Sistema Tipo 3-2: Aceleração

$$ {}_{S_1}^{\,I}\mathbf{a}^P = \begin{Bmatrix} R\ddot{\theta}\ c\theta - R(\dot{\theta}^2 + \dot{\beta}^2)\ s\theta \\ R\ddot{\beta}\ s\theta + 2R\dot{\beta}\dot{\theta}\ c\theta \\ -R\ddot{\theta}\ s\theta - R\dot{\theta}^2\ c\theta \end{Bmatrix} $$

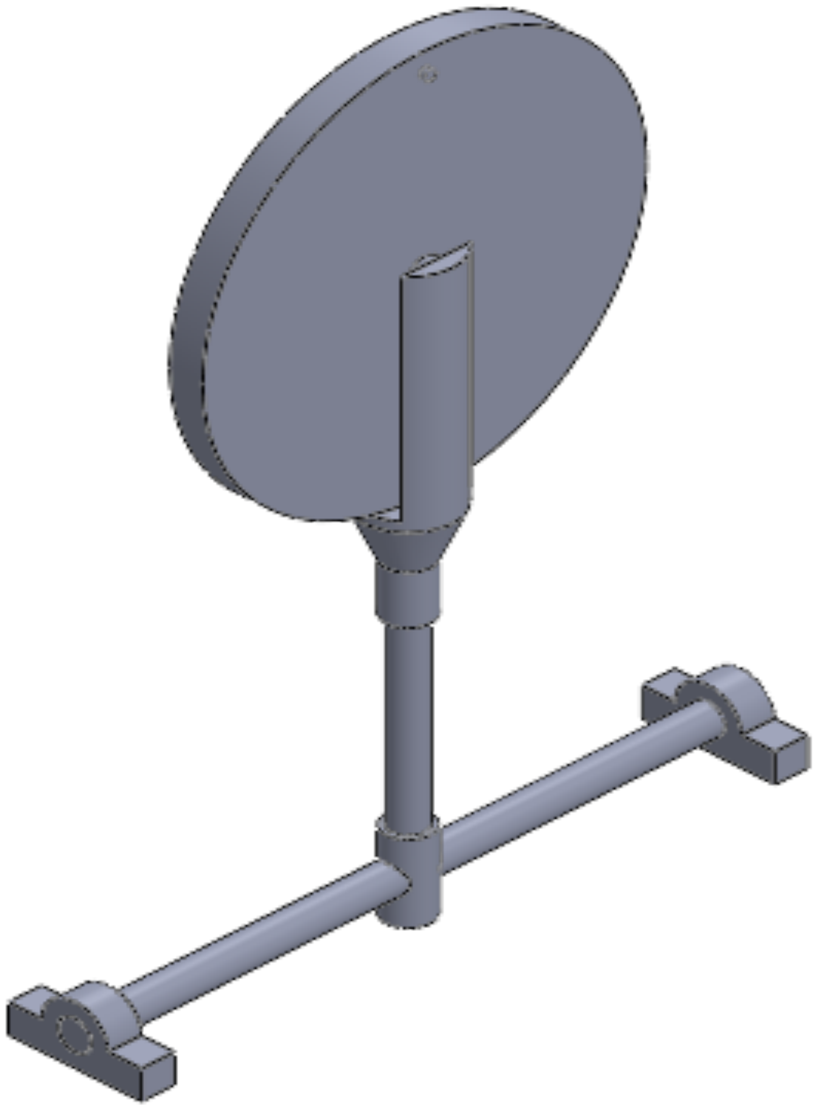
$$ {}_{S_2}^{\,I}\mathbf{a}^P = \mathbf{T}_\theta\, {}_{S_1}^{\,I}\mathbf{a}^P = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \begin{Bmatrix} R\ddot{\theta}\ c\theta - R(\dot{\theta}^2 + \dot{\beta}^2)\ s\theta \\ R\ddot{\beta}\ s\theta + 2R\dot{\beta}\dot{\theta}\ c\theta \\ -R\ddot{\theta}\ s\theta - R\dot{\theta}^2\ c\theta \end{Bmatrix} = \begin{Bmatrix} R\ddot{\theta} \ - R\dot{\beta}^2\ s\theta c\theta \\ R\ddot{\beta}\ s\theta + 2R\dot{\beta}\dot{\theta}\ c\theta \\ R\dot{\theta}^2 - R\dot{\beta}^2 s^2\theta \end{Bmatrix} $$

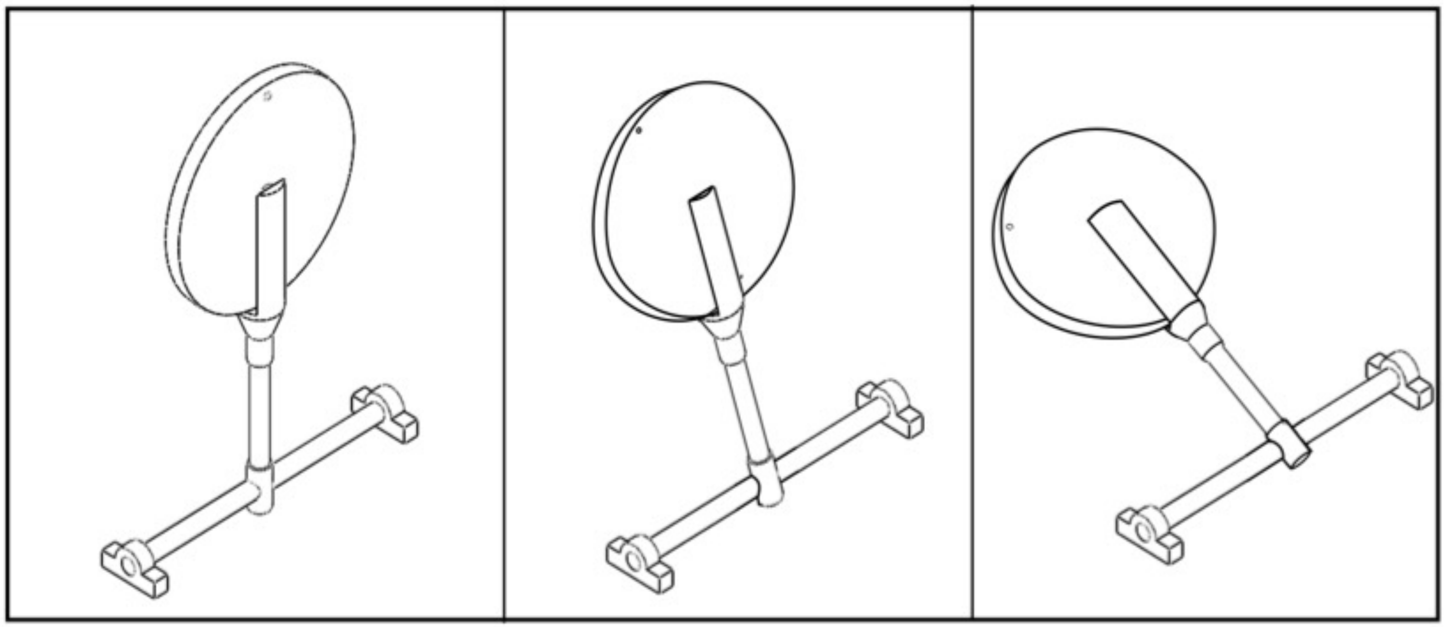




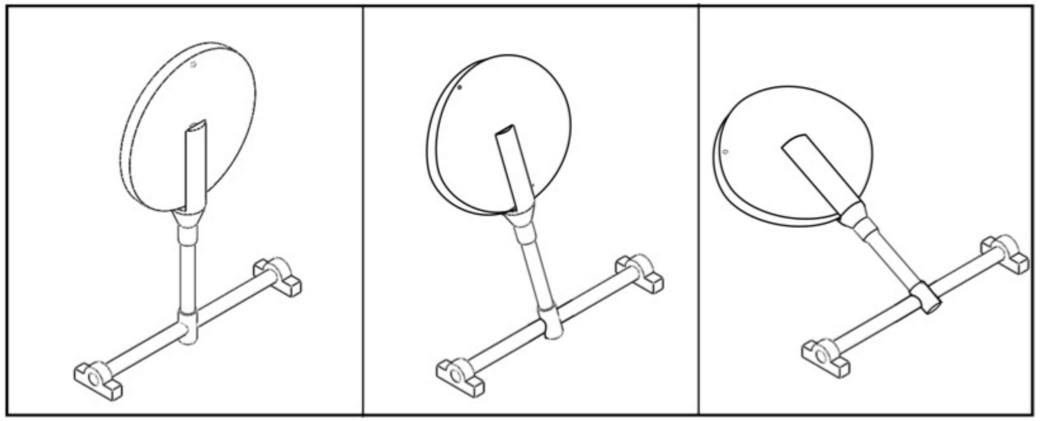


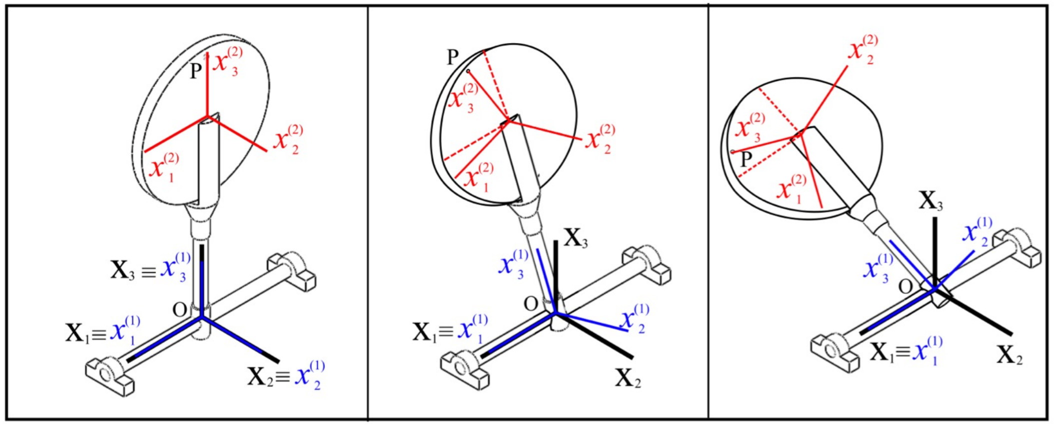
Sistema Tipo 1-2

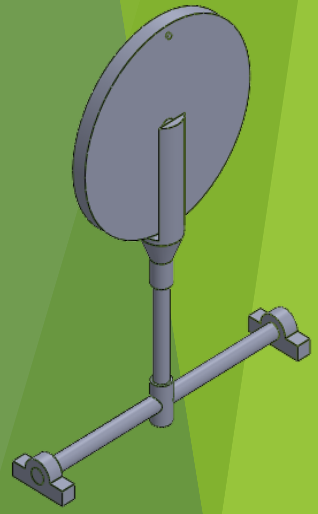


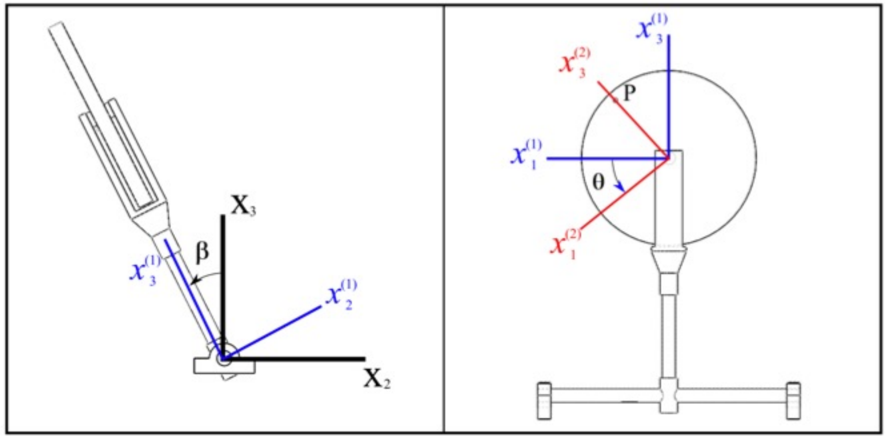


Sistema Tipo 1-2







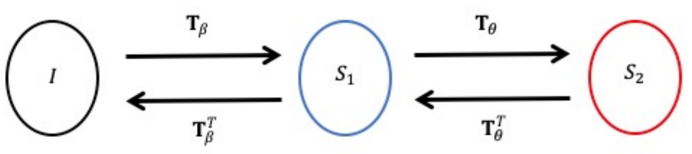


- Inercial, I (X_i)
- Móvel 1, solidário à haste, S_1 ($x_i^{(1)}$)
- Móvel 2, solidário ao disco, S_2 ($x_i^{(2)}$)

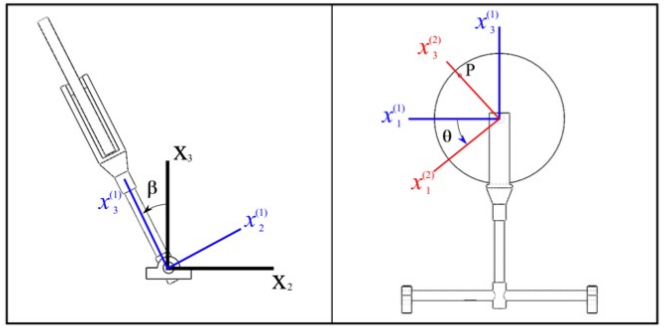
Sistema S_1: $$\mathbf{T}_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\beta & s\beta \\ 0 & -s\beta & c\beta \end{bmatrix}$$

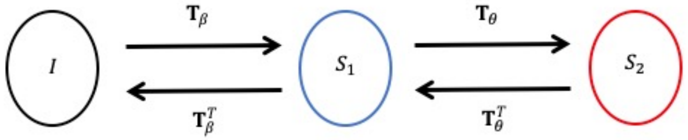
Sistema S_2: $$\mathbf{T}_\theta = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}$$

Tipo 1-2



Sistema Tipo 1-2: Velocidade e Aceleração Angulares



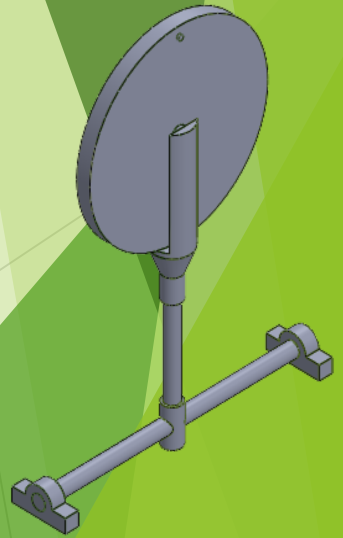


$$ {}^{I}\boldsymbol{\omega}^{S_2} = {}^{I}\boldsymbol{\omega}^{S_1} + {}^{S_1}\boldsymbol{\omega}^{S_2} $$

$$ {}^{I}_{S_1}\boldsymbol{\omega}^{S_1} = \mathbf{T}_\beta \, {}^{I}_{I}\boldsymbol{\omega}^{S_1} = \begin{Bmatrix} \dot\beta \\ 0 \\ 0 \end{Bmatrix} \qquad {}^{S_1}_{S_2}\boldsymbol{\omega}^{S_2} = {}^{S_1}_{S_1}\boldsymbol{\omega}^{S_2} = \begin{Bmatrix} 0 \\ \dot\theta \\ 0 \end{Bmatrix} $$

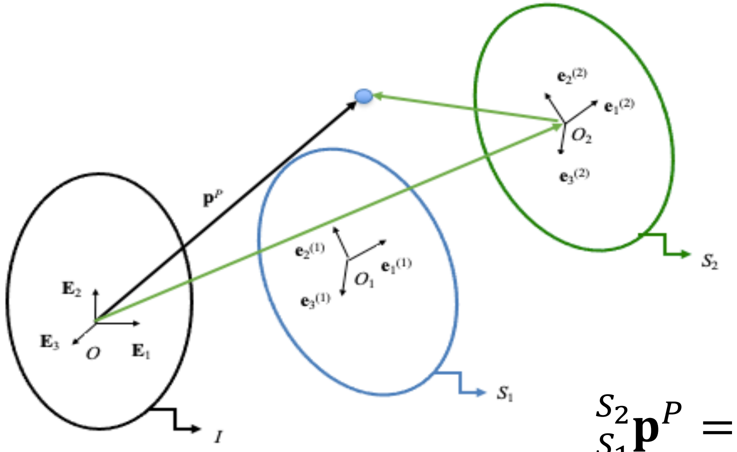
$$ {}^{I}_{S_1}\boldsymbol{\omega}^{S_2} = {}^{I}_{S_1}\boldsymbol{\omega}^{S_1} + {}^{S_1}_{S_1}\boldsymbol{\omega}^{S_2} = \begin{Bmatrix} \dot\beta \\ \dot\theta \\ 0 \end{Bmatrix} $$

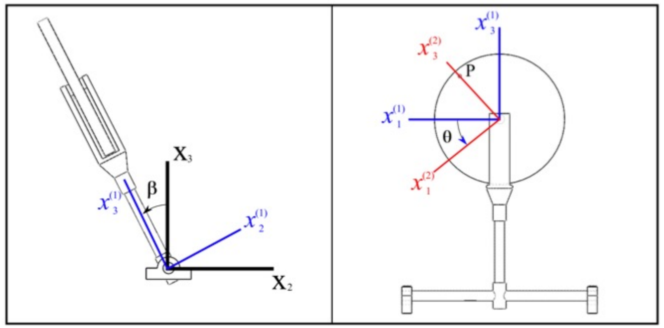
$$ {}^{I}_{S_1}\boldsymbol{\alpha}^{S_2} = {}^{I}_{S_1}\boldsymbol{\alpha}^{S_1} + {}^{S_1}_{S_1}\boldsymbol{\alpha}^{S_2} + {}^{I}_{S_1}\boldsymbol{\omega}^{S_1} \times {}^{S_1}_{S_1}\boldsymbol{\omega}^{S_2} = \begin{Bmatrix} \ddot\beta \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \ddot\theta \\ 0 \end{Bmatrix} + \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ \dot\beta & 0 & 0 \\ 0 & \dot\theta & 0 \end{vmatrix} = \begin{Bmatrix} \ddot\beta \\ \ddot\theta \\ \dot\beta\dot\theta \end{Bmatrix} $$

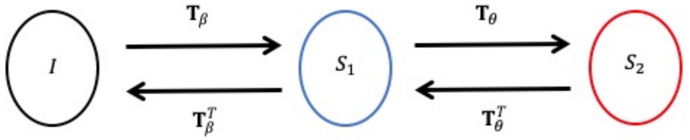


Sistema Tipo 1-2: Posição

- Usando o referencial S_2







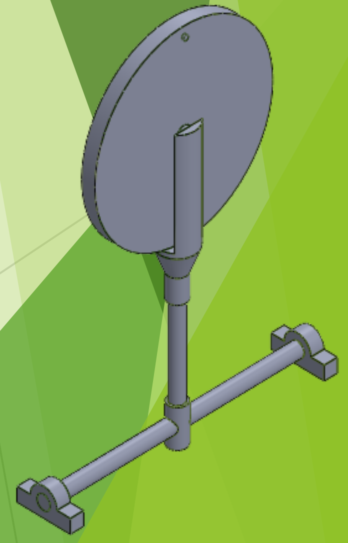
$$ {}^{I}_{S_1}\mathbf{p}^P = {}^{I}_{S_1}\mathbf{p}^{O_2} + {}^{S_2}_{S_1}\mathbf{p}^P $$

$$ {}^{S_2}_{S_1}\mathbf{p}^P = {}^{S_1}_{S_1}\mathbf{p}^P $$

$$ {}^{I}_{S_1}\mathbf{p}^{O_2} = \begin{Bmatrix} 0 \\ 0 \\ L \end{Bmatrix} $$

$$ {}^{S_2}_{S_1}\mathbf{p}^P = \mathbf{T}_\theta^T \; {}^{S_2}_{S_2}\mathbf{p}^P = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ R \end{Bmatrix} = \begin{Bmatrix} R\,s\theta \\ 0 \\ R\,c\theta \end{Bmatrix} $$

$$ {}^{I}_{S_1}\mathbf{p}^P = \begin{Bmatrix} R\,s\theta \\ 0 \\ L + R\,c\theta \end{Bmatrix} $$



Sistema Tipo 1-2: Velocidade

$$ {}_{S_1}^{I}\mathbf{v}^{P} = {}_{S_1}\mathbf{v}^{O_2} + {}_{S_1}^{S_2}\mathbf{v}^{P} + {}_{S_1}^{I}\boldsymbol{\omega}^{S_2} \times {}_{S_1}^{S_2}\mathbf{p}^{P} $$

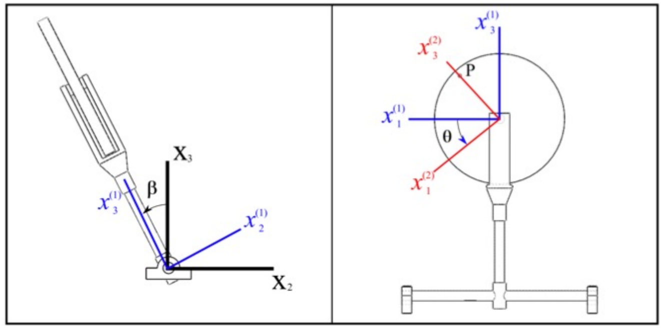
0

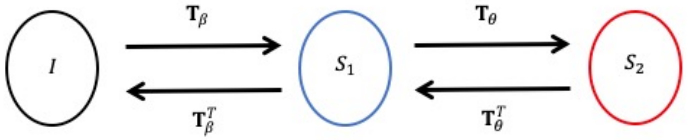
$$ {}_{S_1}^{I}\mathbf{v}^{O_2} = \begin{Bmatrix} 0 \\ -L\,\dot{\beta} \\ 0 \end{Bmatrix} $$

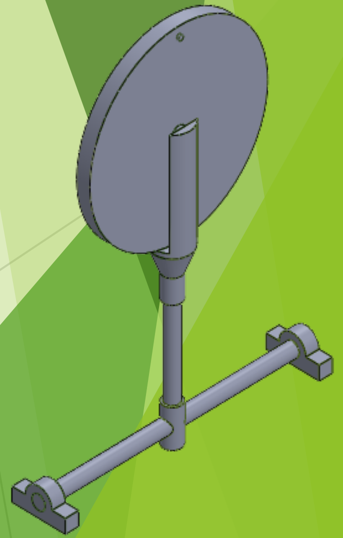
$$ {}_{S_1}^{I}\boldsymbol{\omega}^{S_2} \times {}_{S_1}^{S_2}\mathbf{p}^{P} = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ \dot{\beta} & \dot{\theta} & 0 \\ R\,s\theta & 0 & R\,c\theta \end{vmatrix} = \begin{Bmatrix} R\dot{\theta}\,c\theta \\ -R\dot{\beta}\,c\theta \\ -R\dot{\theta}\,s\theta \end{Bmatrix} $$

$$ {}_{S_1}^{I}\mathbf{v}^{P} = \begin{Bmatrix} R\dot{\theta}\,c\theta \\ -L\,\dot{\beta} - R\dot{\beta}\,c\theta \\ -R\dot{\theta}\,s\theta \end{Bmatrix} $$

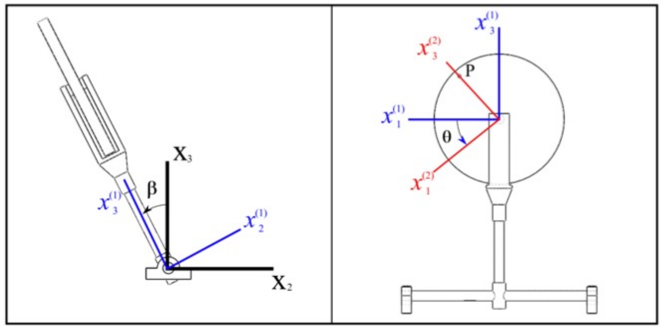
$$ {}_{I}^{I}\mathbf{v}^{P} = \mathbf{T}_{\beta}^{T}\,{}_{S_1}^{I}\mathbf{v}^{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\beta & s\beta \\ 0 & -s\beta & c\beta \end{bmatrix} \begin{Bmatrix} R\dot{\theta}\,c\theta \\ -L\,\dot{\beta} - R\dot{\beta}\,c\theta \\ -R\dot{\theta}\,s\theta \end{Bmatrix} = \begin{Bmatrix} R\dot{\theta}\,c\theta \\ -L\,\dot{\beta}\,c\beta - R\dot{\beta}\,c\theta\,c\beta - R\dot{\theta}\,s\theta\,s\beta \\ L\,\dot{\beta}\,s\beta + R\dot{\beta}\,c\theta\,s\beta - R\dot{\theta}\,s\theta\,c\beta \end{Bmatrix} $$

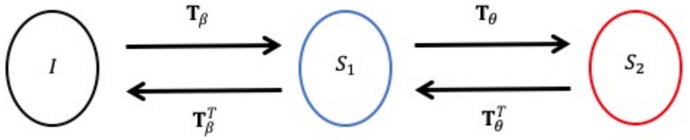






Sistema Tipo 1-2: Aceleração





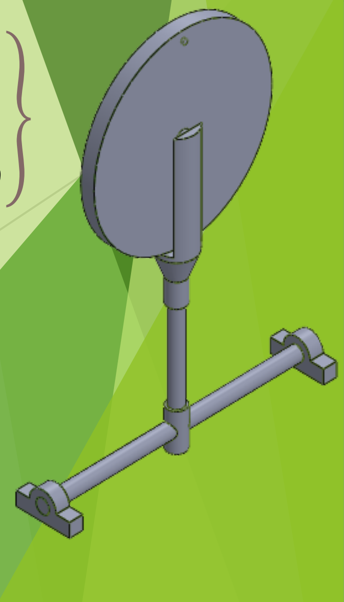
$$ {}^{I}_{S_1}\mathbf{a}^P = {}^{I}_{S_1}\mathbf{a}^{O_2} + \underbrace{{}^{S_2}_{S_1}\mathbf{a}^P}_{0} + {}^{I}_{S_1}\boldsymbol{\alpha}^{S_2}\times{}^{S_2}_{S_1}\mathbf{p}^P + {}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times\left({}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_1}\mathbf{p}^P\right) + \underbrace{2\,{}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_1}\mathbf{v}^P}_{0} $$

$$ {}^{I}_{S_1}\mathbf{a}^{O_2} = \begin{Bmatrix} 0 \\ -L\ddot{\beta} \\ -L\dot{\beta}^2 \end{Bmatrix} $$

$$ {}^{I}_{S_1}\boldsymbol{\alpha}^{S_2}\times{}^{S_2}_{S_1}\mathbf{p}^P = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ \ddot{\beta} & \ddot{\theta} & \dot{\beta}\dot{\theta} \\ R\,s\theta & 0 & R\,c\theta \end{vmatrix} = \begin{Bmatrix} -R\ddot{\theta}\,c\theta \\ -R\ddot{\beta}\,c\theta + R\dot{\beta}\dot{\theta}\,s\theta \\ -R\ddot{\theta}\,s\theta \end{Bmatrix} $$

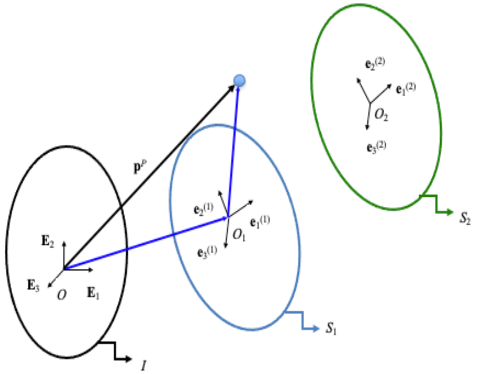
$$ {}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times\left({}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_1}\mathbf{p}^P\right) = {}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times\begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ \dot{\beta} & \dot{\theta} & 0 \\ R\,s\theta & 0 & R\,c\theta \end{vmatrix} = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ \dot{\beta} & \dot{\theta} & 0 \\ R\dot{\theta}\,s\theta & -R\dot{\beta}\,c\theta & -R\dot{\theta}\,c\theta \end{vmatrix} = \begin{Bmatrix} -R\dot{\theta}^2\,c\theta \\ R\dot{\beta}\dot{\theta}\,s\theta \\ -R\dot{\beta}^2\,c\theta - R\dot{\theta}^2\,s\theta \end{Bmatrix} $$

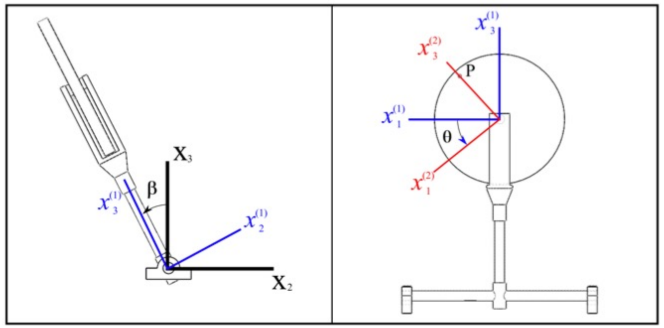
$$ {}^{I}_{S_1}\mathbf{a}^P = \begin{Bmatrix} -R\ddot{\theta}\,c\theta - R\dot{\theta}^2\,c\theta \\ -L\ddot{\beta} - R\ddot{\beta}\,c\theta + 2R\dot{\beta}\dot{\theta}\,s\theta \\ -R\ddot{\theta}\,s\theta - L\dot{\beta}^2 - R\dot{\beta}^2\,c\theta - R\dot{\theta}^2\,s\theta \end{Bmatrix} $$

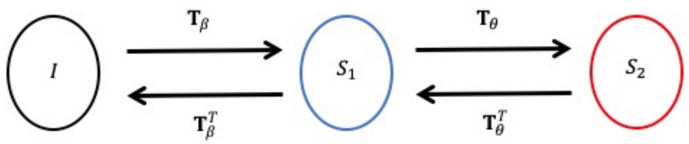


Sistema Tipo 1-2: Aceleração

- Alternativa usando o referencial S_1







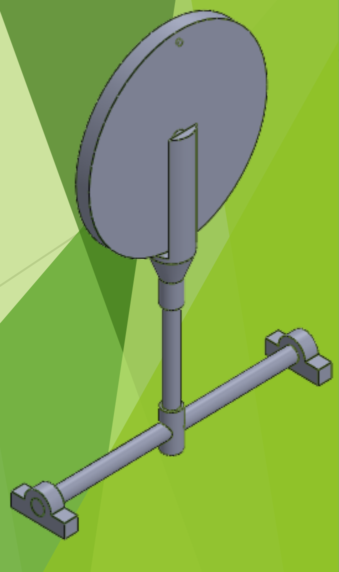
$$ {}_{S_1}^{I}\mathbf{a}^{P} = {}_{S_1}^{I}\mathbf{a}^{O_1} + {}_{S_1}^{S_1}\mathbf{a}^{P} + {}_{S_1}^{I}\boldsymbol{\alpha}^{S_1}\times{}_{S_1}^{S_1}\mathbf{p}^{P} + {}_{S_1}^{I}\boldsymbol{\omega}^{S_1}\times\left({}_{S_1}^{I}\boldsymbol{\omega}^{S_1}\times{}_{S_1}^{S_1}\mathbf{p}^{P}\right) + 2\,{}_{S_1}^{I}\boldsymbol{\omega}^{S_1}\times{}_{S_1}^{S_1}\mathbf{v}^{P} $$

$$ {}_{S_1}^{I}\mathbf{a}^{O_1} = \begin{Bmatrix} 0 \\ -L\ddot{\beta} \\ -L\dot{\beta}^2 \end{Bmatrix} $$

$$ {}_{S_1}^{S_1}\mathbf{a}^{P} = {}_{S_1}^{S_1}\boldsymbol{\alpha}^{S_2}\times{}_{S_1}^{S_1}\mathbf{p}^{P} + {}_{S_1}^{S_1}\boldsymbol{\omega}^{S_2}\times\left({}_{S_1}^{S_1}\boldsymbol{\omega}^{S_2}\times{}_{S_1}^{S_1}\mathbf{p}^{P}\right) = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & \ddot{\theta} & 0 \\ R\,s\theta & 0 & R\,c\theta \end{vmatrix} + \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ 0 & \dot{\theta} & 0 \\ R\dot{\theta}\,s\theta & 0 & -R\dot{\theta}\,c\theta \end{vmatrix} $$

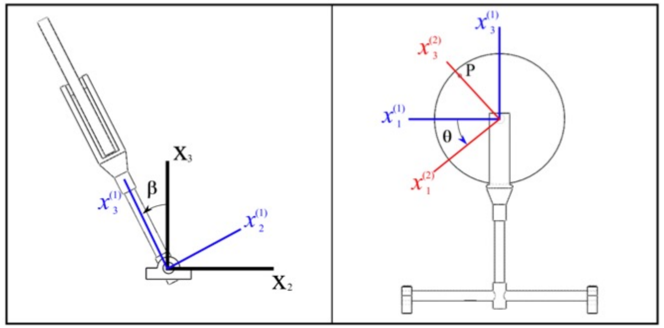
$$ = \begin{Bmatrix} R\ddot{\theta}\,c\theta - R\dot{\theta}^2\,c\theta \\ 0 \\ R\ddot{\theta}\,s\theta - R\dot{\theta}^2\,s\theta \end{Bmatrix} $$

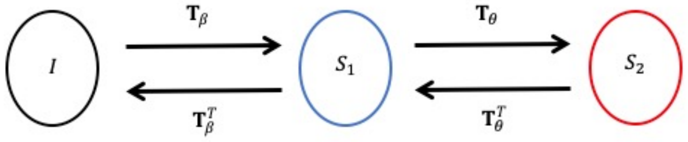
$$ {}_{S_1}^{I}\boldsymbol{\alpha}^{S_1}\times{}_{S_1}^{S_1}\mathbf{p}^{P} = \begin{vmatrix} \mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)} \\ \ddot{\beta} & 0 & 0 \\ R\,s\theta & 0 & R\,c\theta \end{vmatrix} = \begin{Bmatrix} 0 \\ -R\ddot{\beta}\,c\theta \\ 0 \end{Bmatrix} $$



Sistema Tipo 1-2: Aceleração

- **Alternativa usando o referencial** S_1





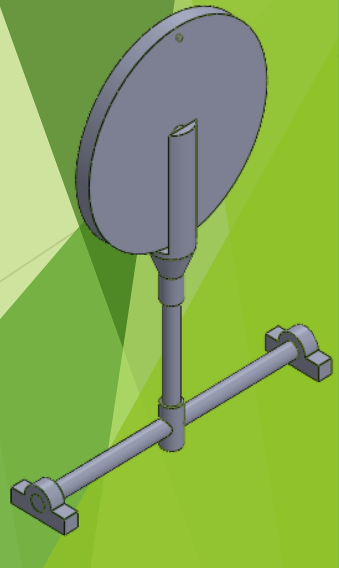
$${}^{I}_{S_1}\mathbf{a}^P = {}^{I}_{S_1}\mathbf{a}^{O_1} + {}^{S_1}_{S_1}\mathbf{a}^P + {}^{I}_{S_1}\boldsymbol{\alpha}^{S_1}\times{}^{S_1}_{S_1}\mathbf{p}^P + {}^{I}_{S_1}\boldsymbol{\omega}^{S_1}\times\left({}^{I}_{S_1}\boldsymbol{\omega}^{S_1}\times{}^{S_1}_{S_1}\mathbf{p}^P\right) + 2\,{}^{I}_{S_1}\boldsymbol{\omega}^{S_1}\times{}^{S_1}_{S_1}\mathbf{v}^P$$

$${}^{I}_{S_1}\boldsymbol{\omega}^{S_1}\times\left({}^{I}_{S_1}\boldsymbol{\omega}^{S_1}\times{}^{S_1}_{S_1}\mathbf{p}^P\right) = {}^{I}_{S_1}\boldsymbol{\omega}^{S_2}\times\begin{vmatrix}\mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)}\\ \dot\beta & 0 & 0\\ R\,s\theta & 0 & R\,c\theta\end{vmatrix} = \begin{vmatrix}\mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)}\\ \dot\beta & 0 & 0\\ 0 & -R\dot\beta\,c\theta & 0\end{vmatrix} = \begin{Bmatrix}0\\ 0\\ -R\dot\beta^2\,c\theta\end{Bmatrix}$$

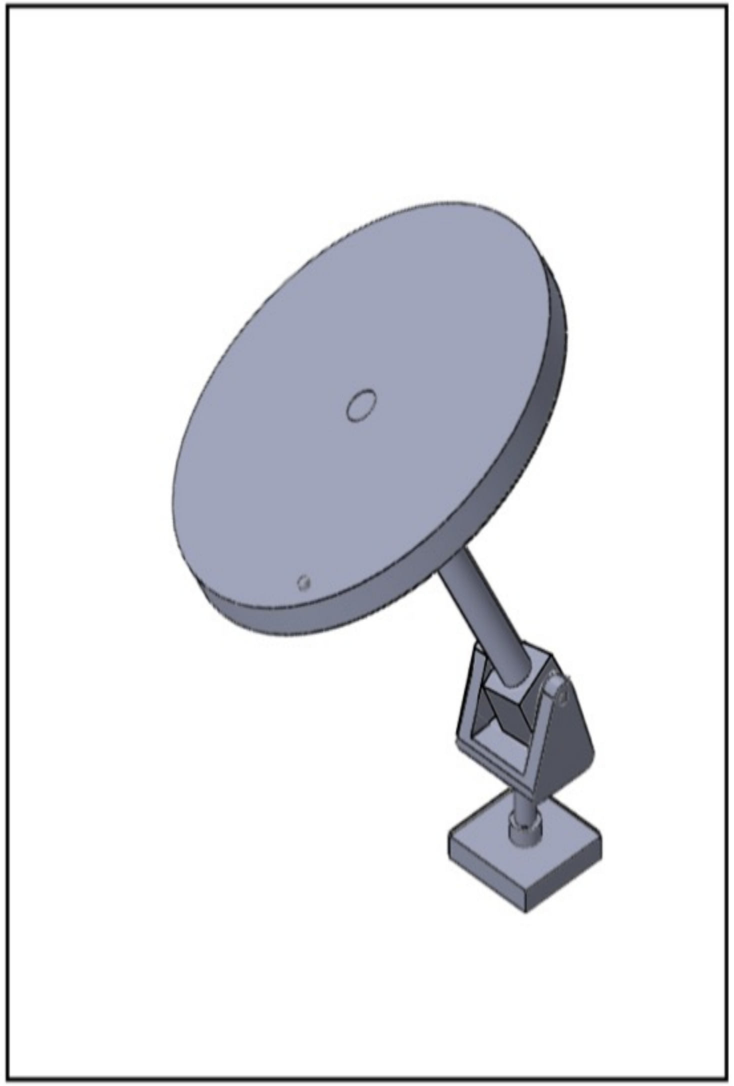
$${}^{S_1}_{S_1}\mathbf{v}^P = \begin{vmatrix}\mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)}\\ 0 & \dot\theta & 0\\ R\,s\theta & 0 & R\,c\theta\end{vmatrix} = \begin{Bmatrix}R\dot\theta\,c\theta\\ 0\\ -R\dot\theta\,s\theta\end{Bmatrix}$$

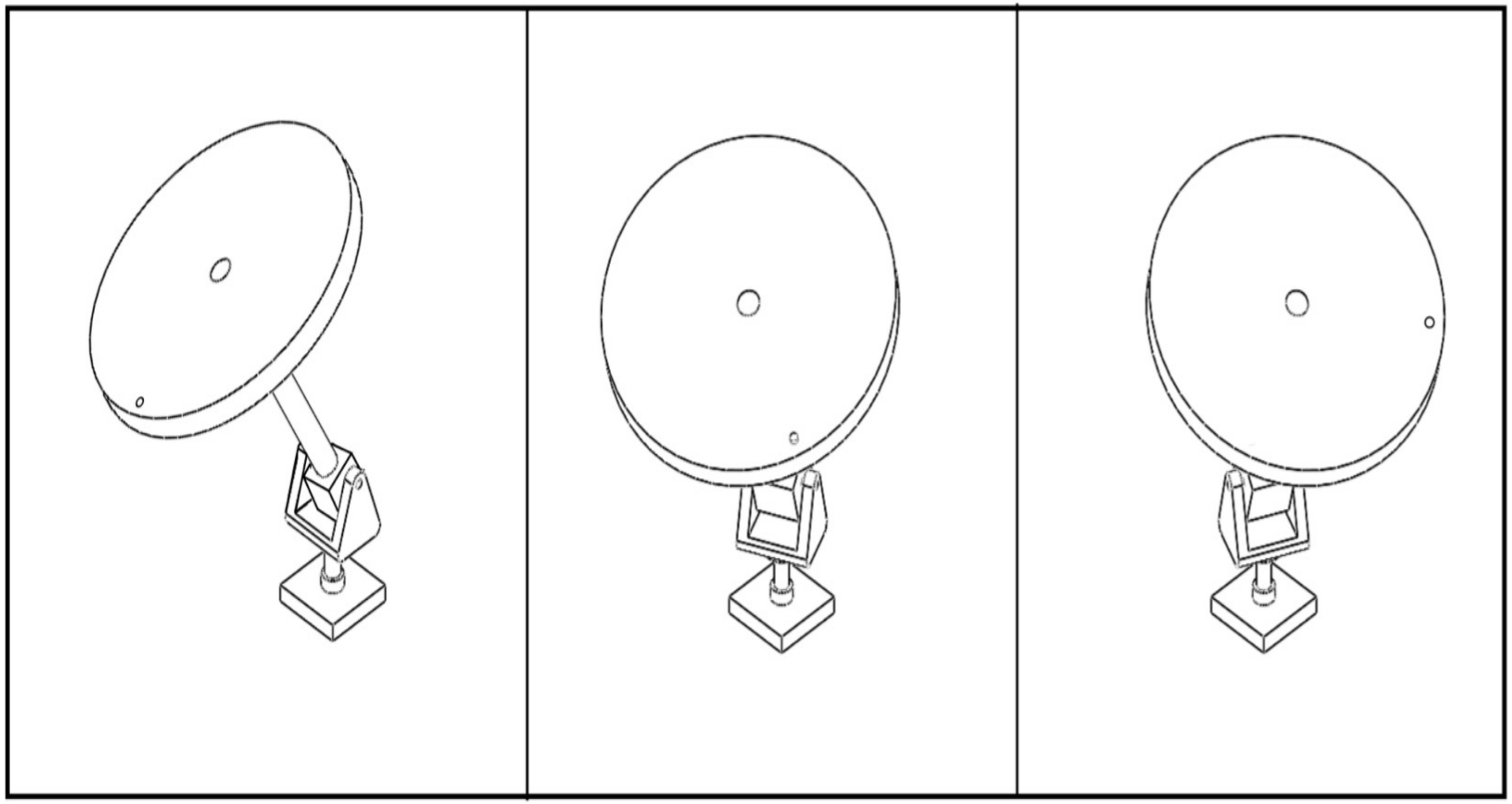
$${}^{I}_{S_1}\mathbf{a}^P = \begin{Bmatrix}-R\ddot\theta\,c\theta - R\dot\theta^2\,c\theta\\ -L\ddot\beta - R\ddot\beta\,c\theta + 2R\dot\beta\dot\theta\,s\theta\\ -R\ddot\theta\,s\theta - L\dot\beta^2 - R\dot\beta^2\,c\theta - R\dot\theta^2\,s\theta\end{Bmatrix}$$

$$2\,{}^{I}_{S_1}\boldsymbol{\omega}^{S_1}\times{}^{S_1}_{S_1}\mathbf{v}^P = 2\begin{vmatrix}\mathbf{e}_1^{(1)} & \mathbf{e}_2^{(1)} & \mathbf{e}_3^{(1)}\\ \dot\beta & 0 & 0\\ R\dot\theta\,c\theta & 0 & -R\dot\theta\,s\theta\end{vmatrix} = \begin{Bmatrix}0\\ 2R\dot\beta\dot\theta\,s\theta\\ 0\end{Bmatrix}$$

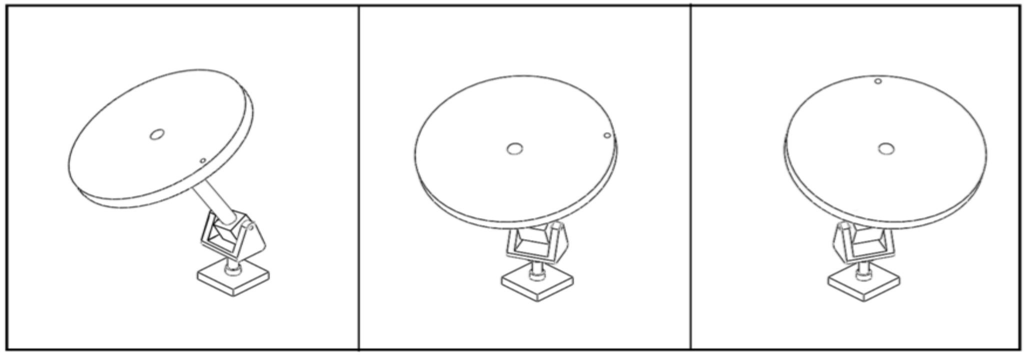


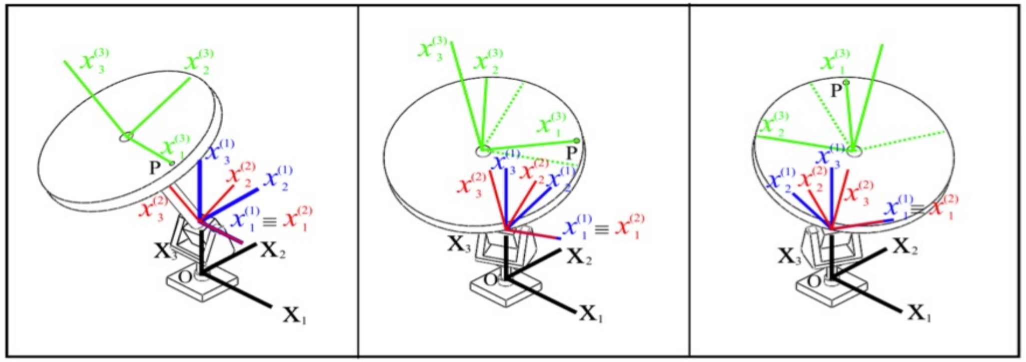
Sistema Tipo 3-(1)-3

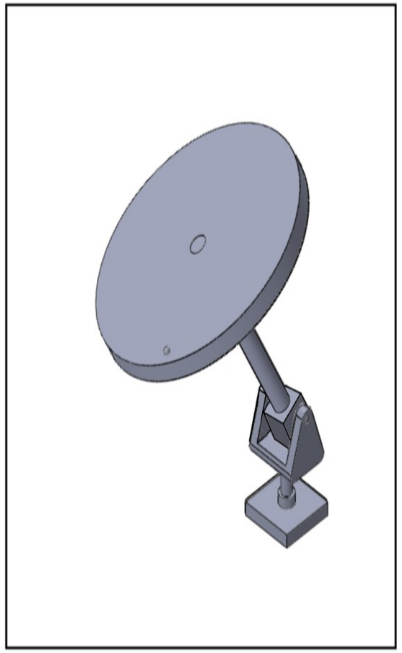


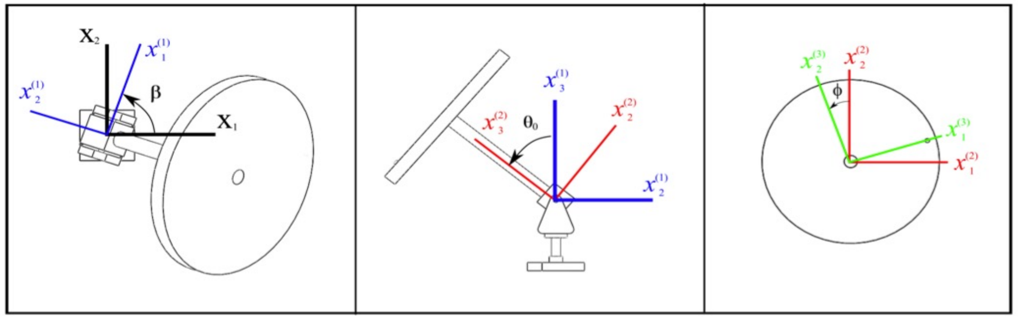


Sistema Tipo 3-(1)-3



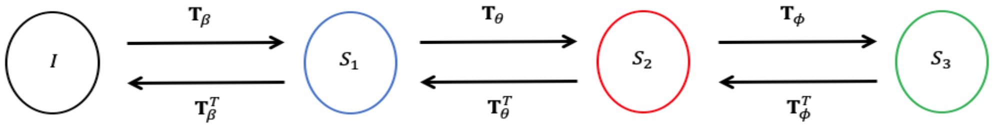






- Inercial, I (X_i)
- Móvel 1, solidário à haste girante na vertical, S_1 ($x_i^{(1)}$)
- Móvel 2, solidário à haste inclinada, S_2 ($x_i^{(2)}$)
- Móvel 3, solidário ao disco, S_3 ($x_i^{(3)}$)

Tipo 3-(1)-3

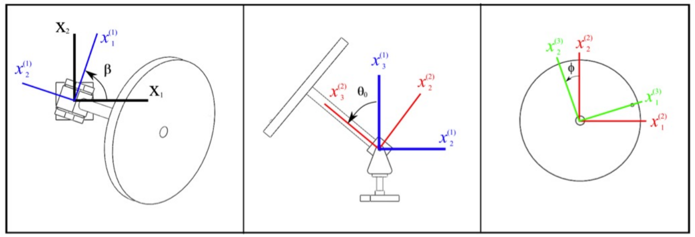


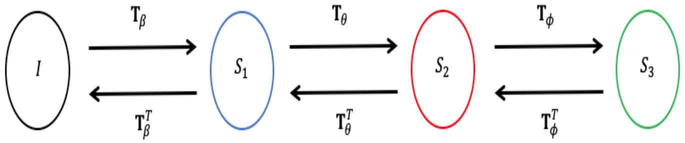
Sistema S_1: $$\mathbf{T}_\beta = \begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sistema S_2: $$\mathbf{T}_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_0 & s\theta_0 \\ 0 & -s\theta_0 & c\theta_0 \end{bmatrix}$$

Sistema S_3: $$\mathbf{T}_\phi = \begin{bmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sistema Tipo 3-(1)-3: Velocidade Angular





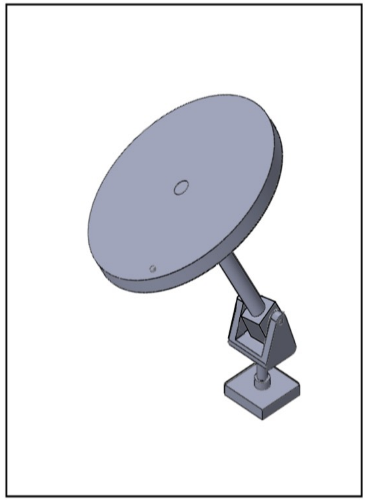
$$^{I}\boldsymbol{\omega}^{S_3} = {}^{I}\boldsymbol{\omega}^{S_1} + {}^{S_1}\boldsymbol{\omega}^{S_2} + {}^{S_2}\boldsymbol{\omega}^{S_3}$$

0

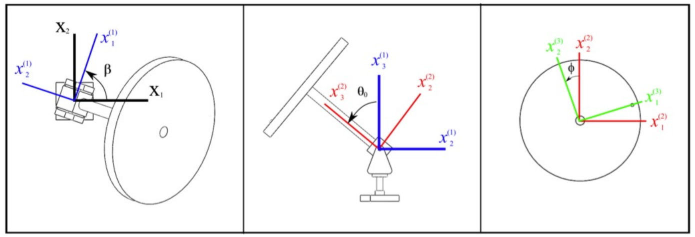
$$_{S_2}^{\;I}\boldsymbol{\omega}^{S_1} = \mathbf{T}_{\theta}\mathbf{T}_{\beta}\,{}_{I}^{I}\boldsymbol{\omega}^{S_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_0 & s\theta_0 \\ 0 & -s\theta_0 & c\theta_0 \end{bmatrix} \begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\beta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \dot{\beta}\, s\theta_0 \\ \dot{\beta}\, c\theta_0 \end{Bmatrix}$$

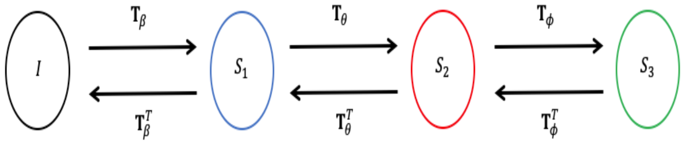
$$_{S_2}^{S_2}\boldsymbol{\omega}^{S_3} = \mathbf{T}_{\phi}^{T} \begin{Bmatrix} 0 \\ 0 \\ \dot{\phi} \end{Bmatrix} = \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\phi} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\phi} \end{Bmatrix}$$

$$_{S_2}^{\;I}\boldsymbol{\omega}^{S_3} = \begin{Bmatrix} 0 \\ \dot{\beta}\, s\theta_0 \\ \dot{\beta}\, c\theta_0 + \dot{\phi} \end{Bmatrix}$$



Sistema Tipo 3-(1)-3: Aceleração Angular

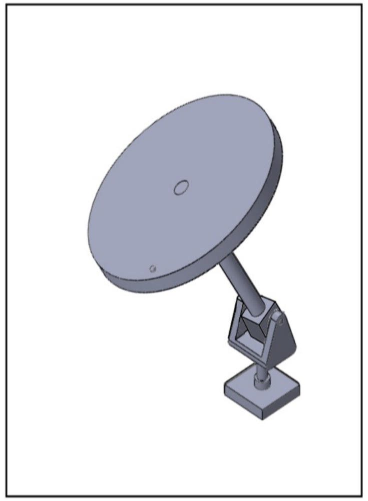




$$ {}^{I}_{S_2}\boldsymbol{\alpha}^{S_3} = {}^{I}_{S_2}\boldsymbol{\alpha}^{S_1} + {}^{S_2}_{S_2}\boldsymbol{\alpha}^{S_3} + {}^{I}_{S_2}\boldsymbol{\omega}^{S_2} \times {}^{S_2}_{S_2}\boldsymbol{\omega}^{S_3} $$

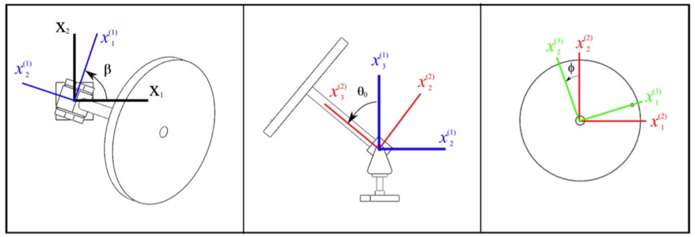
$$ {}^{I}_{S_2}\boldsymbol{\alpha}^{S_3} = {}^{I}_{S_2}\boldsymbol{\alpha}^{S_1} + {}^{S_2}_{S_2}\boldsymbol{\alpha}^{S_3} + {}^{I}_{S_2}\boldsymbol{\omega}^{S_2} \times {}^{S_2}_{S_2}\boldsymbol{\omega}^{S_3} = \begin{Bmatrix} 0 \\ \ddot{\beta}\, s\theta_0 \\ \ddot{\beta}\, c\theta_0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \ddot{\phi} \end{Bmatrix} + \begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ 0 & \dot{\beta}\, s\theta_0 & \dot{\beta}\, c\theta_0 \\ 0 & 0 & \dot{\phi} \end{vmatrix} $$

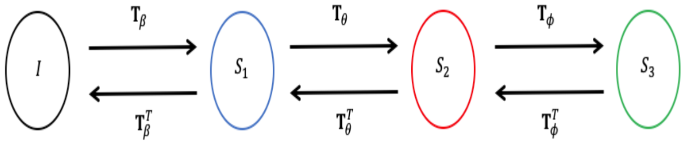
$$ {}^{I}_{S_2}\boldsymbol{\alpha}^{S_3} = \begin{Bmatrix} \dot{\beta}\dot{\phi}\, s\theta_0 \\ \ddot{\beta}\, s\theta_0 \\ \ddot{\beta}\, c\theta_0 + \ddot{\phi} \end{Bmatrix} $$

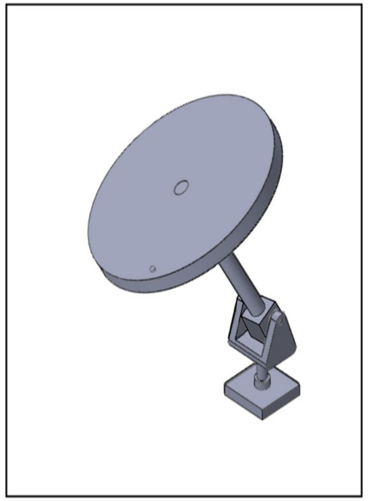


Sistema Tipo 3-(1)-3: Posição

- Usando o referencial S_3







$${}^{I}_{S_2}\mathbf{p}^{P} = {}^{I}_{S_2}\mathbf{p}^{O_3} + {}^{S_3}_{S_2}\mathbf{p}^{P}$$

$${}^{I}_{S_2}\mathbf{p}^{O_3} = \begin{Bmatrix} 0 \\ 0 \\ L \end{Bmatrix}$$

$${}^{S_3}_{S_2}\mathbf{p}^{P} = \mathbf{T}_\phi^T \, {}^{S_3}_{S_3}\mathbf{p}^{P} = \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} R \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} R\,c\phi \\ R\,s\phi \\ 0 \end{Bmatrix}$$

$${}^{I}_{S_2}\mathbf{p}^{P} = \begin{Bmatrix} R\,c\phi \\ R\,s\phi \\ L \end{Bmatrix}$$

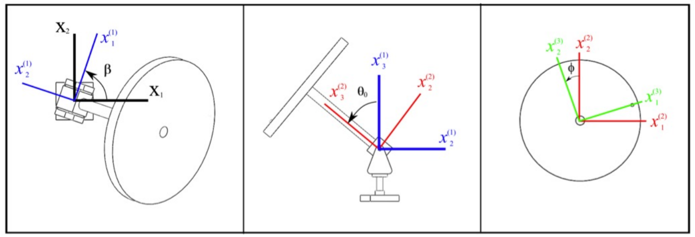
Sistema Tipo 3-(1)-3: Velocidade

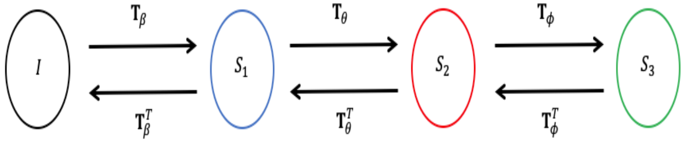
$$ {}^{I}_{S_2}\mathbf{v}^{P} = {}^{I}_{S_2}\mathbf{v}^{O_3} + \underbrace{{}^{S_3}_{S_2}\mathbf{v}^{P}}_{0} + {}^{I}_{S_2}\boldsymbol{\omega}^{S_3}\times{}^{S_3}_{S_1}\mathbf{p}^{P} $$

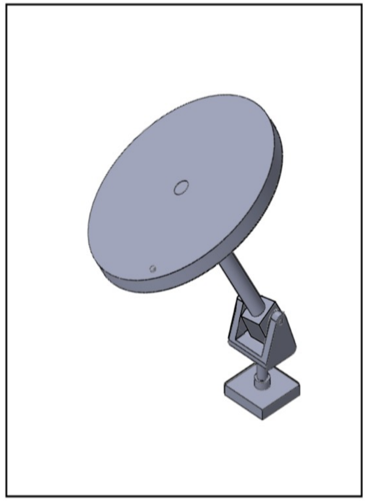
$$ {}^{I}_{S_2}\mathbf{v}^{O_3} = {}^{I}_{S_2}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_2}\mathbf{p}^{O_3} = \begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ 0 & \dot{\beta}\; s\theta_0 & \dot{\beta}\; c\theta_0 \\ 0 & 0 & L \end{vmatrix} = \begin{Bmatrix} L\dot{\beta}\; s\theta_0 \\ 0 \\ 0 \end{Bmatrix} $$

$$ {}^{I}_{S_2}\boldsymbol{\omega}^{S_3}\times{}^{S_3}_{S_2}\mathbf{p}^{P} = \begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ 0 & \dot{\beta}\; s\theta_0 & \dot{\beta}\; c\theta_0 + \dot{\phi} \\ R\; c\phi & R\; s\phi & 0 \end{vmatrix} = \begin{Bmatrix} -(R\dot{\beta}\; c\theta_0 + R\dot{\phi})\; s\phi \\ (R\dot{\beta}\; c\theta_0 + R\dot{\phi})\; c\phi \\ -R\dot{\beta}\; s\theta_0 c\phi \end{Bmatrix} $$

$$ {}^{I}_{S_2}\mathbf{v}^{P} = \begin{Bmatrix} -(R\dot{\beta}\; c\theta_0 + R\dot{\phi})s\phi + L\dot{\beta}\; s\theta_0 \\ (R\dot{\beta}\; c\theta_0 + R\dot{\phi})\; c\phi \\ -R\dot{\beta}\; s\theta_0 c\phi \end{Bmatrix} $$







Sistema Tipo 3-(1)-3: Velocidade

- Alternativa usando o referencial S_2

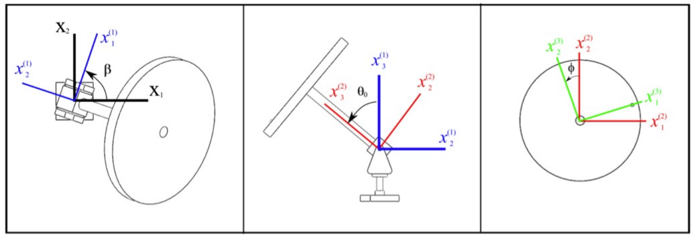
$${}^{I}_{S_2}\mathbf{v}^{P} = {}_{S_2}\mathbf{v}^{O_2} + {}^{S_2}_{S_2}\mathbf{v}^{P} + {}^{I}_{S_2}\boldsymbol{\omega}^{S_2} \times {}^{S_2}_{S_1}\mathbf{p}^{P}$$

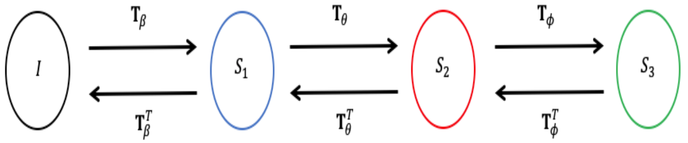
0

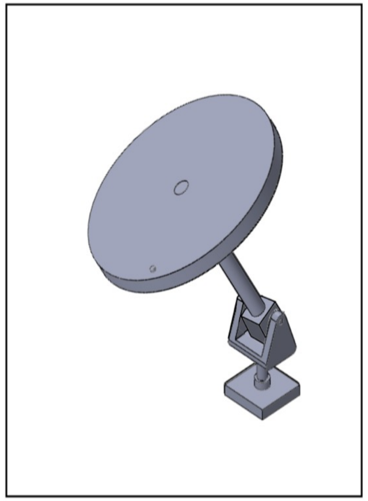
$${}^{S_2}_{S_2}\mathbf{v}^{P} = \mathbf{T}_{\phi}^{T}\,{}^{S_2}_{S_3}\mathbf{v}^{P} = \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ R\dot{\phi} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -R\dot{\phi}\, s\phi \\ R\dot{\phi}\, c\phi \\ 0 \end{Bmatrix}$$

$${}^{I}_{S_2}\boldsymbol{\omega}^{S_2} \times {}^{S_2}_{S_2}\mathbf{p}^{P} = \begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ 0 & \dot{\beta}\, s\theta_0 & \dot{\beta}\, c\theta_0 \\ R\, c\phi & R\, s\phi & L \end{vmatrix} = \begin{Bmatrix} -R\dot{\beta}\, c\theta_0 s\phi + L\dot{\beta}\, s\theta_0 \\ R\dot{\beta}\, c\theta_0\, c\phi \\ -R\dot{\beta}\, s\theta_0 c\phi \end{Bmatrix}$$

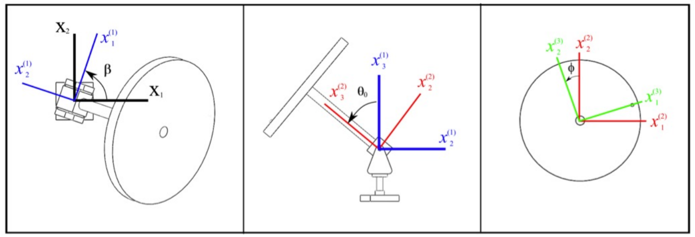
$${}^{I}_{S_2}\mathbf{v}^{P} = \begin{Bmatrix} -\left(R\dot{\beta}\, c\theta_0 + R\dot{\phi}\right) s\phi + L\dot{\beta}\, s\theta_0 \\ \left(R\dot{\beta}\, c\theta_0 + R\dot{\phi}\right) c\phi \\ -R\dot{\beta}\, s\theta_0 c\phi \end{Bmatrix}$$

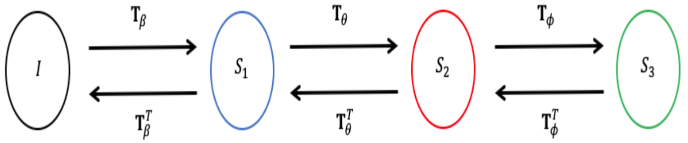






Sistema Tipo 3-(1)-3: Aceleração



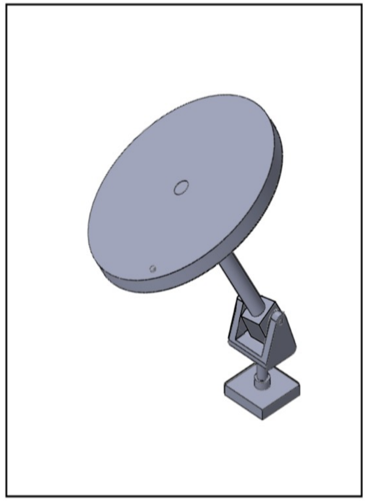


- Usando o referencial S_3

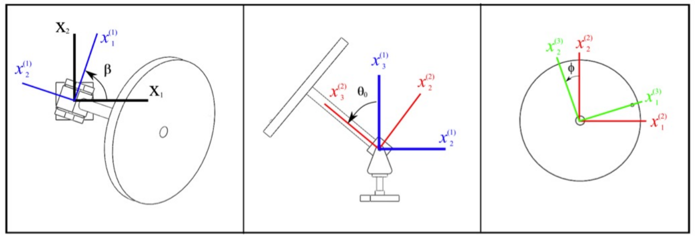
$$
{}^{I}_{S_2}\mathbf{a}^{P} = {}^{I}_{S_2}\mathbf{a}^{O_3} + \underbrace{{}^{S_3}_{S_2}\mathbf{a}^{P}}_{0} + {}^{I}_{S_2}\boldsymbol{\alpha}^{S_3}\times{}^{S_3}_{S_2}\mathbf{p}^{P} + {}^{I}_{S_2}\boldsymbol{\omega}^{S_3}\times\left({}^{I}_{S_2}\boldsymbol{\omega}^{S_3}\times{}^{S_3}_{S_2}\mathbf{p}^{P}\right) + \underbrace{2\,{}^{I}_{S_2}\boldsymbol{\omega}^{S_3}\times{}^{S_3}_{S_2}\mathbf{v}^{P}}_{0}
$$

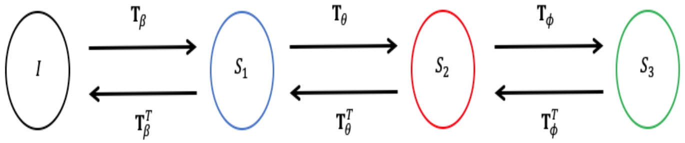
$$
\begin{aligned}
{}^{I}_{S_2}\mathbf{a}^{O_3} &= {}^{I}_{S_2}\boldsymbol{\alpha}^{S_2}\times{}^{S_2}_{S_2}\mathbf{p}^{O_3} + {}^{I}_{S_2}\boldsymbol{\omega}^{S_2}\times\left({}^{I}_{S_2}\boldsymbol{\omega}^{S_2}\times{}^{S_2}_{S_2}\mathbf{p}^{O_3}\right) \\
&= \begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ 0 & \ddot{\beta}\, s\theta_0 & \ddot{\beta}\, s\theta_0 \\ 0 & 0 & L \end{vmatrix} + {}^{I}_{S_2}\boldsymbol{\omega}^{S_2}\times\begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ 0 & \dot{\beta}\, s\theta_0 & \dot{\beta}\, c\theta_0 \\ 0 & 0 & L \end{vmatrix} \\
&= \begin{Bmatrix} L\ddot{\beta}\, s\theta_0 \\ 0 \\ 0 \end{Bmatrix} + \begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ 0 & \dot{\beta}\, s\theta_0 & \dot{\beta}\, c\theta_0 \\ L\dot{\beta}\, s\theta_0 & 0 & 0 \end{vmatrix} = \begin{Bmatrix} -L\ddot{\beta}\, s\theta_0 \\ L\dot{\beta}^2\, s\theta_0 c\theta_0 \\ -L\dot{\beta}^2\, s^2\theta_0 \end{Bmatrix}
\end{aligned}
$$

$$
{}^{I}_{S_2}\boldsymbol{\alpha}^{S_3}\times{}^{S_3}_{S_2}\mathbf{p}^{P} = \begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ \dot{\beta}\dot{\phi}\, s\theta_0 & \ddot{\beta}\, s\theta_0 & \ddot{\beta}\, c\theta_0 + \ddot{\phi} \\ R\, c\phi & R\, s\phi & 0 \end{vmatrix} = \begin{Bmatrix} -\left(R\ddot{\beta}\, c\theta_0 + R\ddot{\phi}\right) s\phi \\ \left(R\ddot{\beta}\, c\theta_0 + R\ddot{\phi}\right) c\phi \\ -R\ddot{\beta}\, s\theta_0 c\phi + R\dot{\beta}\dot{\phi}\, s\theta_0 s\phi \end{Bmatrix}
$$



Sistema Tipo 3-(1)-3: Aceleração



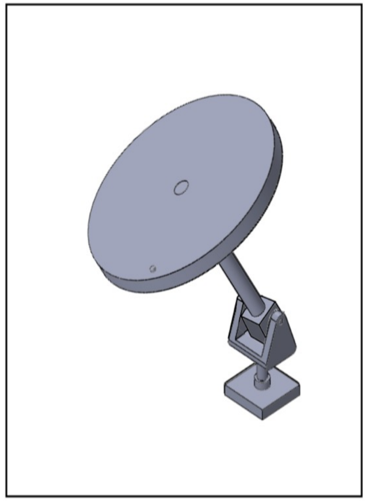


$$
{}^{I}_{S_2}\mathbf{a}^{P} = {}^{I}_{S_2}\mathbf{a}^{O_3} + \underbrace{{}^{S_3}_{S_2}\mathbf{a}^{P}}_{0} + {}^{I}_{S_2}\boldsymbol{\alpha}^{S_3}\times{}^{S_3}_{S_2}\mathbf{p}^{P} + {}^{I}_{S_2}\boldsymbol{\omega}^{S_3}\times\left({}^{I}_{S_2}\boldsymbol{\omega}^{S_3}\times{}^{S_3}_{S_2}\mathbf{p}^{P}\right) + \underbrace{2\,{}^{I}_{S_2}\boldsymbol{\omega}^{S_3}\times{}^{S_3}_{S_2}\mathbf{v}^{P}}_{0}
$$

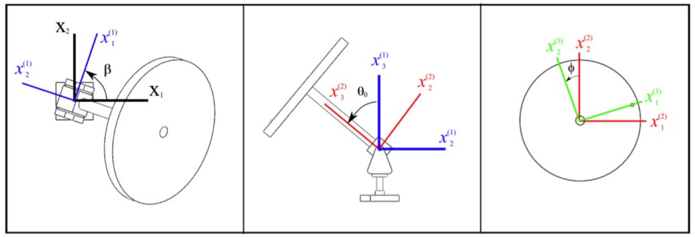
$$
{}^{I}_{S_2}\boldsymbol{\omega}^{S_3}\times\left({}^{I}_{S_2}\boldsymbol{\omega}^{S_3}\times{}^{S_3}_{S_2}\mathbf{p}^{P}\right) = {}^{I}_{S_2}\boldsymbol{\omega}^{S_3}\times\begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ 0 & \dot{\beta}\, s\theta_0 & \dot{\beta}\, c\theta_0 + \dot{\phi} \\ R\, c\phi & R\, s\phi & 0 \end{vmatrix}
$$

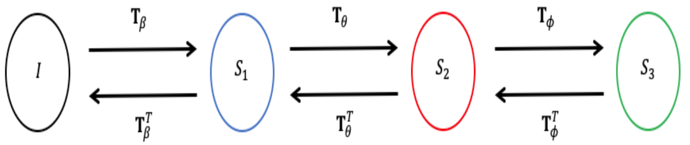
$$
{}^{I}_{S_2}\boldsymbol{\omega}^{S_3}\times\left({}^{I}_{S_2}\boldsymbol{\omega}^{S_3}\times{}^{S_3}_{S_2}\mathbf{p}^{P}\right) = \begin{vmatrix} \mathbf{e}_1^{(2)} & \mathbf{e}_2^{(2)} & \mathbf{e}_3^{(2)} \\ 0 & \dot{\beta}\, s\theta_0 & \dot{\beta}\, c\theta_0 + \dot{\phi} \\ -(R\dot{\beta}\, c\theta_0 + R\dot{\phi})\, s\phi & (R\dot{\beta}\, c\theta_0 + R\dot{\phi})\, c\phi & -R\dot{\beta}\, s\theta_0 c\phi \end{vmatrix}
$$

$$
{}^{I}_{S_2}\boldsymbol{\omega}^{S_3}\times\left({}^{I}_{S_2}\boldsymbol{\omega}^{S_3}\times{}^{S_3}_{S_2}\mathbf{p}^{P}\right) = \begin{Bmatrix} -(R\dot{\beta}^2 + R\dot{\phi}^2 + 2R\dot{\beta}\dot{\phi}\, c\theta_0)c\phi \\ -(R\dot{\beta}^2\, c^2\theta_0 + R\dot{\phi}^2 + 2R\dot{\beta}\dot{\phi}\, c\theta_0)s\phi \\ (R\dot{\beta}^2\, c\theta_0 s\theta_0 + R\dot{\beta}\dot{\phi}\, s\theta_0)s\phi \end{Bmatrix}
$$



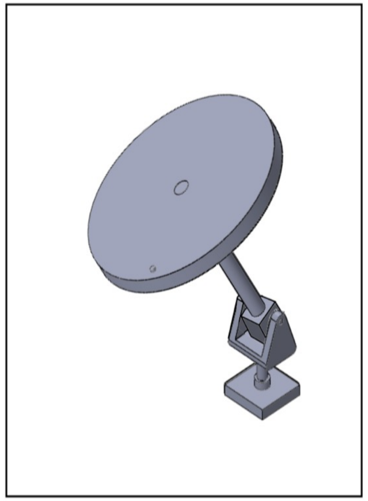
Sistema Tipo 3-(1)-3: Aceleração





$$ {}^I_{S_2}\mathbf{a}^P = {}^I_{S_2}\mathbf{a}^{O_3} + \underbrace{{}^{S_3}_{S_2}\mathbf{a}^P}_{0} + {}^I_{S_2}\boldsymbol{\alpha}^{S_3}\times{}^{S_3}_{S_2}\mathbf{p}^P + {}^I_{S_2}\boldsymbol{\omega}^{S_3}\times\left({}^I_{S_2}\boldsymbol{\omega}^{S_3}\times{}^{S_3}_{S_2}\mathbf{p}^P\right) + \underbrace{2\,{}^I_{S_2}\boldsymbol{\omega}^{S_3}\times{}^{S_3}_{S_2}\mathbf{v}^P}_{0} $$

$$ {}^I_{S_2}\mathbf{a}^P = \begin{Bmatrix} L\ddot{\beta}\, s\theta_0 - \left(R\ddot{\beta}\, c\theta_0 + R\ddot{\phi}\right) s\phi - (R\dot{\beta}^2 + R\dot{\phi}^2 + 2R\dot{\beta}\dot{\phi}\, c\theta_0)c\phi \\ L\dot{\beta}^2\, s\theta_0 c\theta_0 + \left(R\ddot{\beta}\, c\theta_0 + R\ddot{\phi}\right) c\phi - (R\dot{\beta}^2\, c^2\theta_0 + R\dot{\phi}^2 + 2R\dot{\beta}\dot{\phi}\, c\theta_0)s\phi \\ -L\dot{\beta}^2\, s^2\theta_0 - R\ddot{\beta}\, s\theta_0 c\phi + (R\dot{\beta}^2\, c\theta_0 s\theta_0 + 2R\dot{\beta}\dot{\phi}\, s\theta_0)s\phi \end{Bmatrix} $$



$$ {}^I_{S_3}\mathbf{a}^P = \mathbf{T}_\phi\, {}^I_{S_2}\mathbf{a}^P $$

$$ = \begin{bmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -L\ddot{\beta}\, s\theta_0 - \left(R\ddot{\beta}\, c\theta_0 + R\ddot{\phi}\right) s\phi - (R\dot{\beta}^2 + R\dot{\phi}^2 + 2R\dot{\beta}\dot{\phi}\, c\theta_0)c\phi \\ L\dot{\beta}^2\, s\theta_0 c\theta_0 + \left(R\ddot{\beta}\, c\theta_0 + R\ddot{\phi}\right) c\phi - (R\dot{\beta}^2\, c^2\theta_0 + R\dot{\phi}^2 + 2R\dot{\beta}\dot{\phi}\, c\theta_0)s\phi \\ -L\dot{\beta}^2\, s^2\theta_0 - R\ddot{\beta}\, s\theta_0 c\phi + (R\dot{\beta}^2\, c\theta_0 s\theta_0 + 2R\dot{\beta}\dot{\phi}\, s\theta_0)s\phi \end{Bmatrix} $$

$$ = \begin{Bmatrix} L\ddot{\beta}\, s\theta_0 c\phi + L\dot{\beta}^2\, s\theta_0 c\theta_0 s\phi - [R\dot{\beta}^2(1 + c^2\theta_0) + R\dot{\phi}^2 + 2R\dot{\beta}\dot{\phi}\, c\theta_0]c\phi \\ -L\ddot{\beta}\, s\theta_0 s\phi + L\dot{\beta}^2\, s\theta_0 c\theta_0 c\phi + R\ddot{\beta}\, c\theta_0 + R\ddot{\phi} + R\dot{\beta}^2\, s^2\theta_0 s\phi c\phi \\ -L\dot{\beta}^2\, s^2\theta_0 - R\ddot{\beta}\, s\theta_0 c\phi + (R\dot{\beta}^2\, c\theta_0 s\theta_0 + 2R\dot{\beta}\dot{\phi}\, s\theta_0)s\phi \end{Bmatrix} $$