DINÂMICA: Análise Tensorial

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Análise Tensorial

Tensores são entes matemáticos utilizados para representar grandezas físicas.

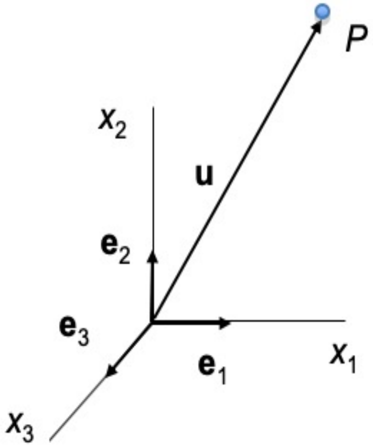
- ***Tensor de ordem zero - escalar***: caracterizado por apenas uma componente.

  Massa e temperatura.
- ***Tensor de ordem um - vetor***: caracterizado por três componentes.

  Força, posição, velocidade, aceleração, quantidade de movimento.
- ***Tensor de ordem dois***: caracterizado por nove componentes.

  Tensão, deformação, inércia.
- ***Tensor de ordem N***: caracterizado 3^N componentes.

Vetores



$$\mathbf{u} = (u_1, u_2, u_3) = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = u_1\,\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$$

- Notação indicial: $\mathbf{u} \equiv u_i\mathbf{e}_i := \sum_{i=1}^{3} u_i\mathbf{e}_i$

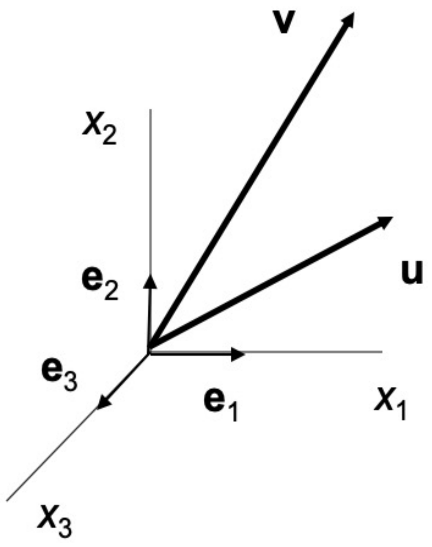
Convenção soma

Índices repetidos, chamados ***mudos***, indicam somatório.

Índices não-repetidos, chamados ***livres***, assumem os valores 1, 2, 3.

Nessa notação, é inconsistente utilizar mais de dois índices mudos por termo.

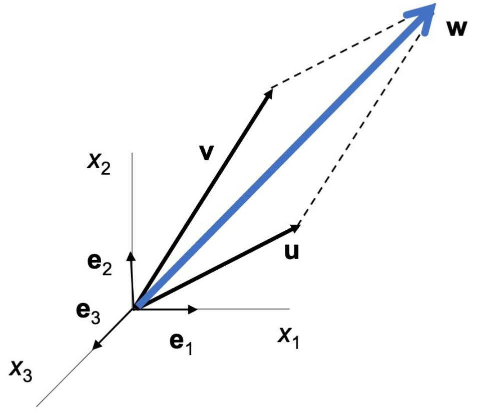
Operação com Vetores

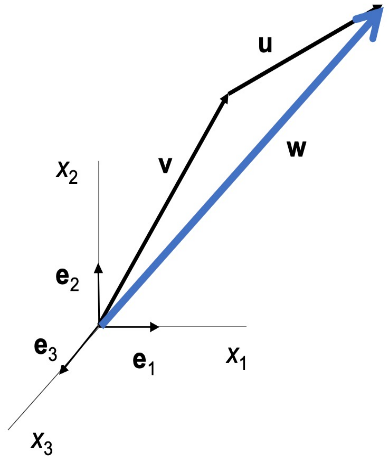


- Soma

$$\mathbf{w} = \mathbf{u} + \mathbf{v}$$

$$w_i = u_i + v_i$$

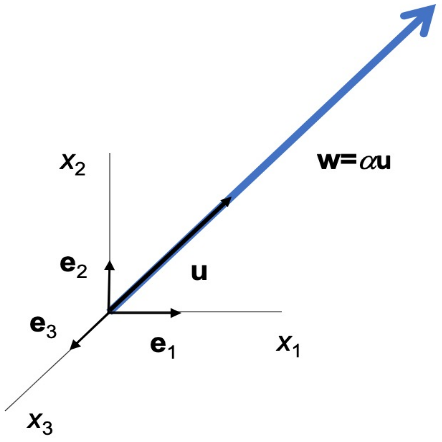




- Produto por um escalar

$$\mathbf{w} = \alpha\,\mathbf{u}$$

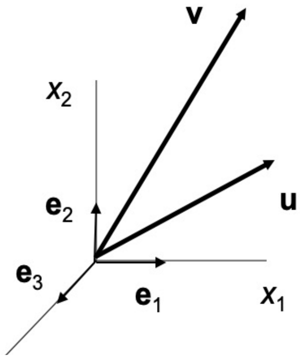
$$w_i = \alpha\, u_i$$



Operação com Vetores

- Produto escalar

$$\mathbf{u}\cdot\mathbf{v} = u_i\mathbf{e}_i \cdot v_j\mathbf{e}_j = u_i v_j(\mathbf{e}_i \cdot \mathbf{e}_j) = u_i v_j \delta_{ij}$$

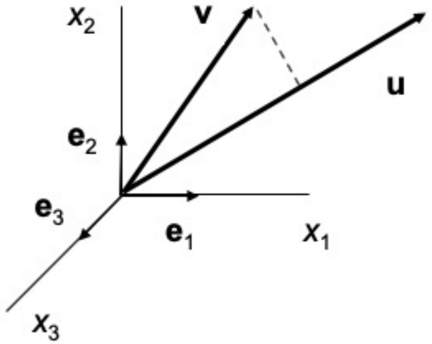


$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

Tensor *Delta de Kronecker*

$$\delta_{ij} = \begin{cases} 1, & \text{se } i = j \\ 0, & \text{se } i \neq j \end{cases}$$

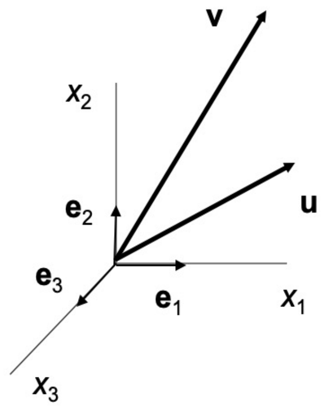
$$u_i v_j \delta_{ij} = u_1v_1\delta_{11} + u_1v_2\delta_{12} + u_1v_3\delta_{13} + u_2v_1\delta_{21} + u_2v_2\delta_{22} + u_2v_3\delta_{23} + u_3v_1\delta_{31} + u_3v_2\delta_{32} + u_3v_3\delta_{33}$$



$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = u_iv_i = u_jv_j$$

Operação com Vetores

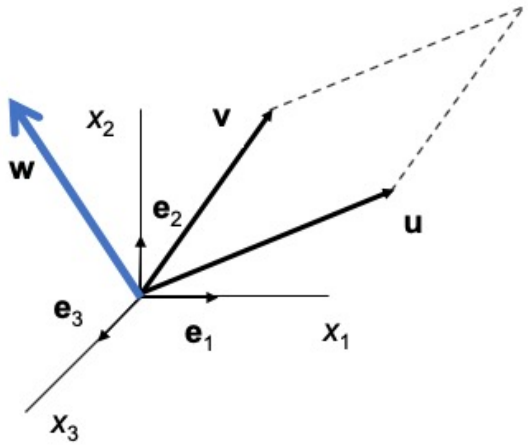
- Produto vetorial



$$\mathbf{w} = \mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{Bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{Bmatrix}$$

$$\mathbf{w} = \mathbf{u}\times\mathbf{v} = u_i\mathbf{e}_i\times v_j\mathbf{e}_j = u_iv_j\left(\mathbf{e}_i\times\mathbf{e}_j\right)$$

$$\mathbf{e}_i\times\mathbf{e}_j = \xi_{ijk}\mathbf{e}_k$$



Tensor *Permutação*

$$\xi_{ijk} = \begin{cases} 0, & \text{se houver índices iguais } (112, 121, 233, \dots) \\ +1, & \text{se houver uma permutação par } (123, 312, 231, \dots) \\ -1, & \text{se houver uma permutação impar } (132, 321, 213, \dots) \end{cases}$$

$$\mathbf{w} = \mathbf{u}\times\mathbf{v} = \xi_{ijk}u_iv_j\mathbf{e}_k$$

Operação com Vetores

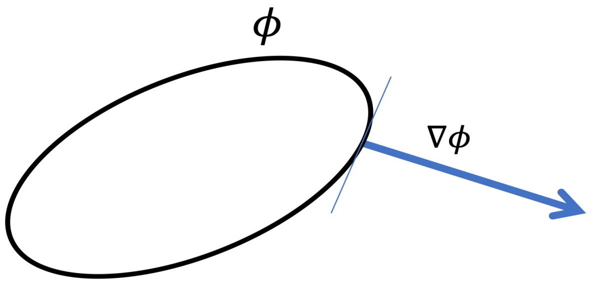
- Produto tensorial

$$\mathbf{u}\otimes\mathbf{v} = \begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix} = u_iv_j$$

Operadores

- Operador Nabla: $\nabla = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3} \equiv \mathbf{e}_i \frac{\partial}{\partial x_i}$

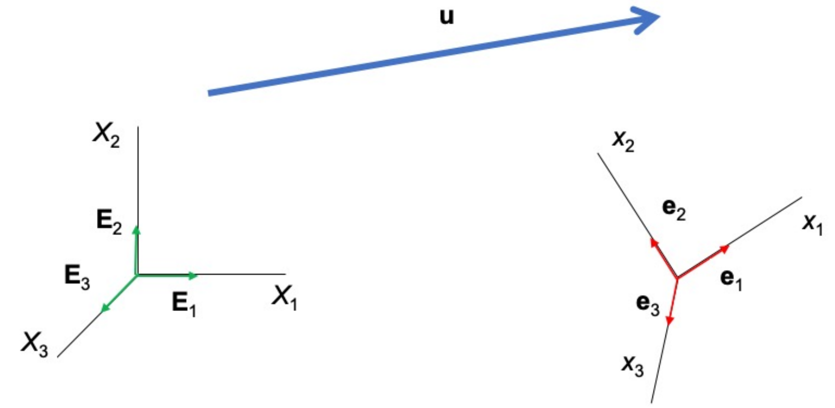
- Gradiente: $\mathrm{grad}(\phi) = \nabla\phi \equiv \mathbf{e}_i \frac{\partial \phi}{\partial x_i} = \mathbf{e}_1 \frac{\partial \phi}{\partial x_1} + \mathbf{e}_2 \frac{\partial \phi}{\partial x_2} + \mathbf{e}_3 \frac{\partial \phi}{\partial x_3}$



- Divergente: $\mathrm{div}(\mathbf{u}) = \nabla.\mathbf{u} \equiv \mathbf{e}_i \frac{\partial}{\partial x_i}.\mathbf{u} = \mathbf{e}_i \frac{\partial}{\partial x_i}.u_j\mathbf{e}_j = \frac{\partial u_j}{\partial x_i}\mathbf{e}_i.\mathbf{e}_j = \frac{\partial u_j}{\partial x_i}\delta_{ij} = \frac{\partial u_i}{\partial x_i}$

- Rotacional: $\mathrm{rot}(\mathbf{u}) = \nabla\times\mathbf{u} \equiv \mathbf{e}_i \frac{\partial}{\partial x_i}\times\mathbf{u} = \mathbf{e}_i \frac{\partial}{\partial x_i}\times u_j\mathbf{e}_j = \frac{\partial u_j}{\partial x_i}\mathbf{e}_i\times\mathbf{e}_j = \xi_{ijk}\frac{\partial u_j}{\partial x_i}\mathbf{e}_k$

Transformação de Coordenadas



$$\mathbf{u} \equiv U_i \mathbf{E}_i = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}$$

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$$\mathbf{u} \equiv U_i \mathbf{E}_i = u_i \mathbf{e}_i$$

$$U_i \mathbf{E}_i \cdot \mathbf{e}_j = u_i \mathbf{e}_i \cdot \mathbf{e}_j$$

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

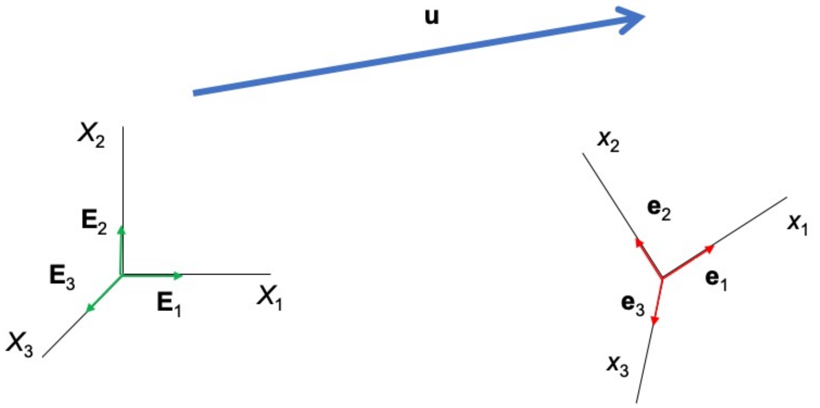
$$\mathbf{E}_i \cdot \mathbf{e}_j = \cos(\mathbf{E}_i, \mathbf{e}_j) = Q_{ij} = T_{ji} = \begin{bmatrix} \mathbf{E}_1 \cdot \mathbf{e}_1 & \mathbf{E}_1 \cdot \mathbf{e}_2 & \mathbf{E}_1 \cdot \mathbf{e}_3 \\ \mathbf{E}_2 \cdot \mathbf{e}_1 & \mathbf{E}_2 \cdot \mathbf{e}_2 & \mathbf{E}_2 \cdot \mathbf{e}_3 \\ \mathbf{E}_3 \cdot \mathbf{e}_1 & \mathbf{E}_3 \cdot \mathbf{e}_2 & \mathbf{E}_3 \cdot \mathbf{e}_3 \end{bmatrix}$$

$$u_j = Q_{ij} U_i = T_{ji} U_i$$

$$\{u\} = [Q]^T \{U\} = [T]\{U\}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} \mathbf{E}_1 \cdot \mathbf{e}_1 & \mathbf{E}_1 \cdot \mathbf{e}_2 & \mathbf{E}_1 \cdot \mathbf{e}_3 \\ \mathbf{E}_2 \cdot \mathbf{e}_1 & \mathbf{E}_2 \cdot \mathbf{e}_2 & \mathbf{E}_2 \cdot \mathbf{e}_3 \\ \mathbf{E}_3 \cdot \mathbf{e}_1 & \mathbf{E}_3 \cdot \mathbf{e}_2 & \mathbf{E}_3 \cdot \mathbf{e}_3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}$$

Transformação de Coordenadas



$$\mathbf{u} \equiv U_i \mathbf{E}_i = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}$$

$$\mathbf{u} \equiv u_i \mathbf{e}_i = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\mathbf{u} \equiv U_i \mathbf{E}_i = u_i \mathbf{e}_i$$

$$U_i \mathbf{E}_i \cdot \mathbf{E}_j = u_i \mathbf{e}_i \cdot \mathbf{E}_j$$

$$\mathbf{E}_i \cdot \mathbf{E}_j = \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

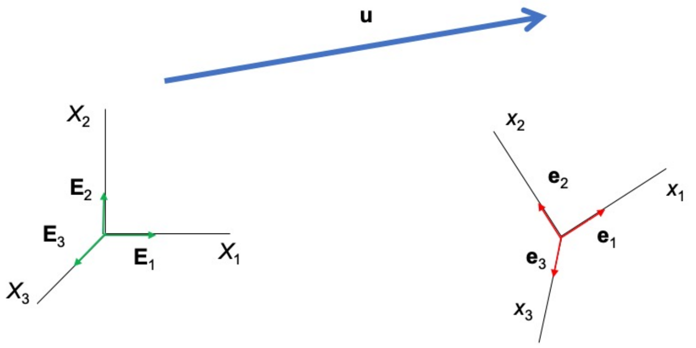
$$\mathbf{e}_i \cdot \mathbf{E}_j = \cos(\mathbf{e}_i, \mathbf{E}_j) = Q_{ji} = T_{ij} = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{E}_1 & \mathbf{e}_1 \cdot \mathbf{E}_2 & \mathbf{e}_1 \cdot \mathbf{E}_3 \\ \mathbf{e}_2 \cdot \mathbf{E}_1 & \mathbf{e}_2 \cdot \mathbf{E}_2 & \mathbf{e}_2 \cdot \mathbf{E}_3 \\ \mathbf{e}_3 \cdot \mathbf{E}_1 & \mathbf{e}_3 \cdot \mathbf{E}_2 & \mathbf{e}_3 \cdot \mathbf{E}_3 \end{bmatrix}$$

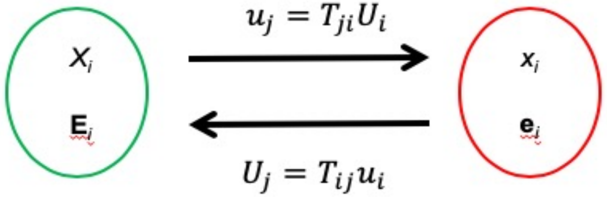
$$U_j = Q_{ji} u_i = T_{ij} u_i$$

$$\{U\} = [Q]\{u\} = [T]^T\{u\}$$

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{E}_1 & \mathbf{e}_1 \cdot \mathbf{E}_2 & \mathbf{e}_1 \cdot \mathbf{E}_3 \\ \mathbf{e}_2 \cdot \mathbf{E}_1 & \mathbf{e}_2 \cdot \mathbf{E}_2 & \mathbf{e}_2 \cdot \mathbf{E}_3 \\ \mathbf{e}_3 \cdot \mathbf{E}_1 & \mathbf{e}_3 \cdot \mathbf{E}_2 & \mathbf{e}_3 \cdot \mathbf{E}_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Transformação de Coordenadas





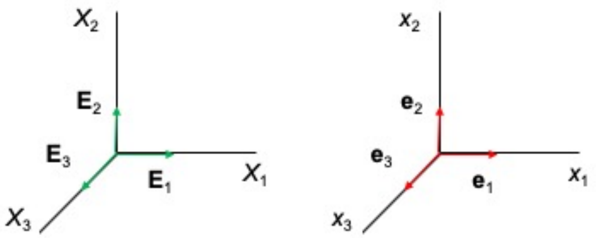
$$\{u\} = [T]\{U\}$$

$$u_j = T_{ji}U_i$$

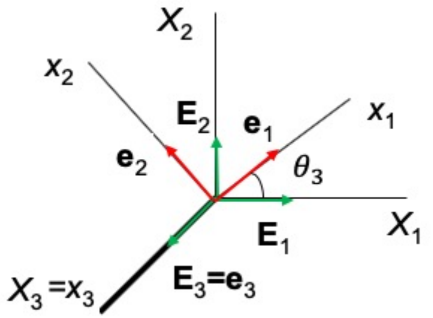
$$U_j = T_{ij}u_i$$

$$\{U\} = [T]^T\{u\}$$

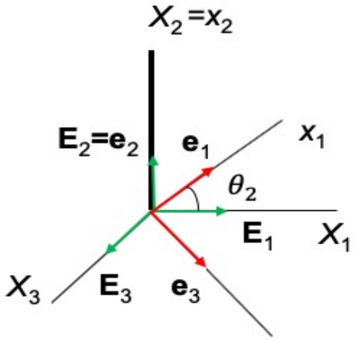
Translação	*Rotações*		



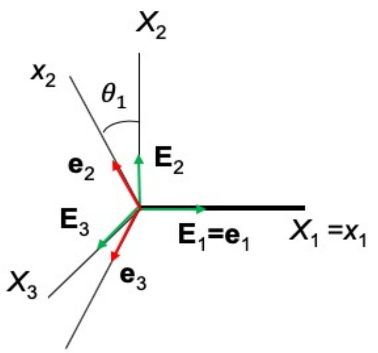
$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\mathbf{T}_{\theta_3} = \begin{bmatrix} c\theta_3 & s\theta_3 & 0 \\ -s\theta_3 & c\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

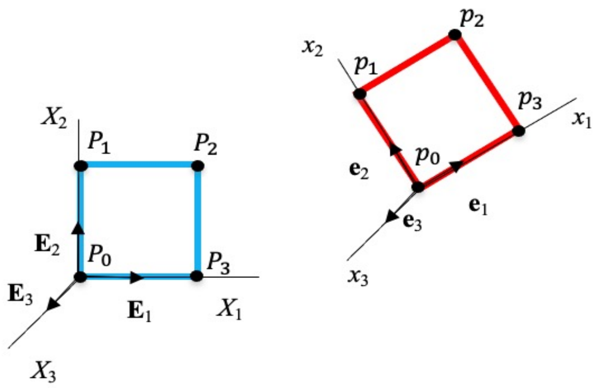


$$\mathbf{T}_{\theta_2} = \begin{bmatrix} c\theta_2 & 0 & -s\theta_2 \\ 0 & 1 & 0 \\ s\theta_2 & 0 & c\theta_2 \end{bmatrix}$$



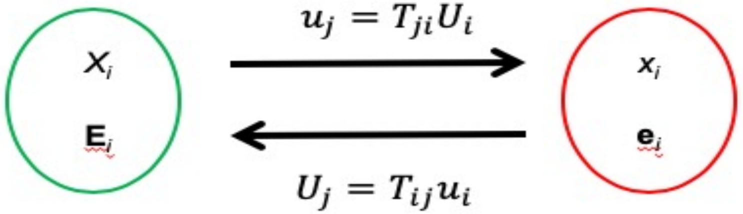
$$\mathbf{T}_{\theta_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_1 & s\theta_1 \\ 0 & -s\theta_1 & c\theta_1 \end{bmatrix}$$

Mapeamentos



$$\{u\} = [T]\{U\}$$

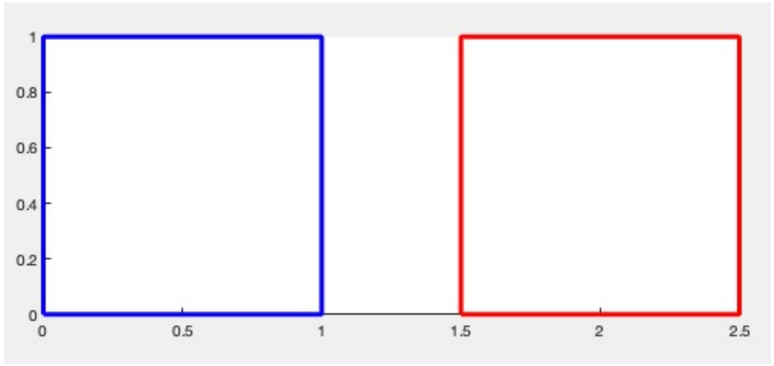
$$u_j = T_{ji} U_i$$



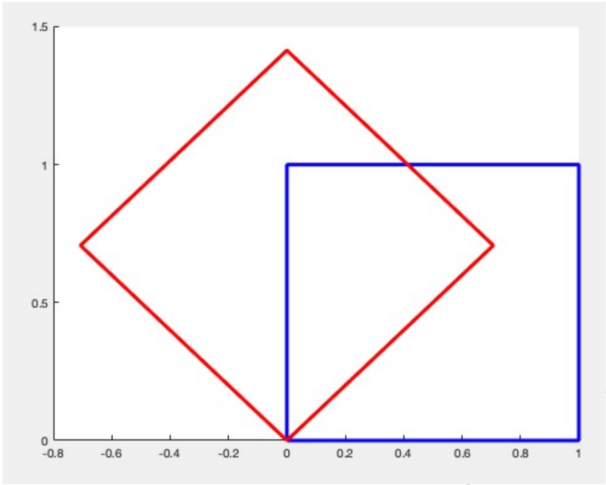
$$U_j = T_{ij} u_i$$

$$\{U\} = [T]^T \{u\}$$

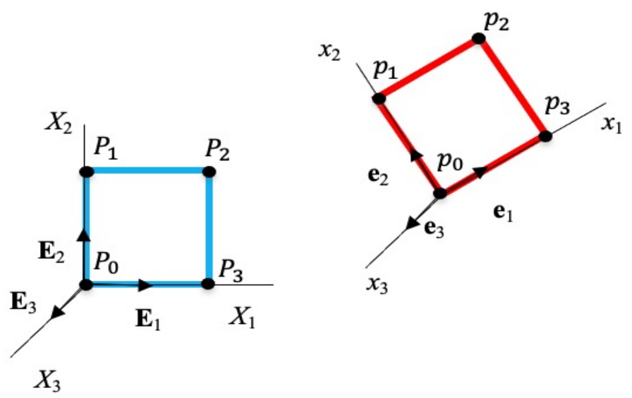
Translação



Rotações

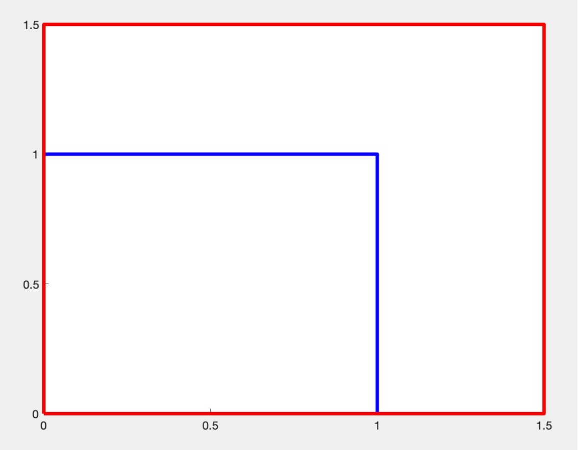


Mapeamentos



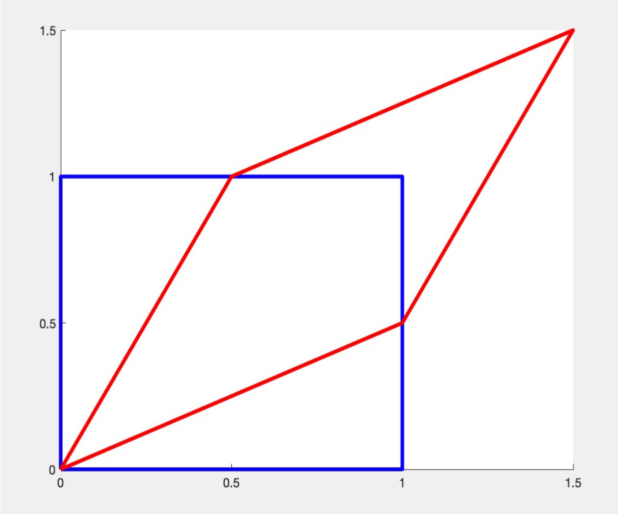
Expansão

$$\mathbf{T} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$



Cisalhamento

$$\mathbf{T} = \begin{bmatrix} 1 & b & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Diferenciação de vetores

$$\mathbf{u} = U_i \mathbf{E}_i \quad \Rightarrow \quad \frac{{}^E d\mathbf{u}}{dt} = \frac{{}^E dU_i}{dt}\mathbf{E}_i = \dot{U}_i \mathbf{E}_i$$

$$\mathbf{u} = u_i \mathbf{e}_i \quad \Rightarrow \quad \frac{{}^E d\mathbf{u}}{dt} = \frac{{}^E du_i}{dt}\mathbf{e}_i + u_i \frac{{}^E d\mathbf{e}_i}{dt} = \dot{u}_i \mathbf{e}_i + u_i \dot{\mathbf{e}}_i$$

$$\frac{{}^E du_i}{dt}\mathbf{e}_i = \frac{{}^e d\mathbf{u}}{dt}$$

$$\frac{{}^E d\mathbf{e}_i}{dt} = \frac{{}^E d(T_{ij}\mathbf{E}_j)}{dt} = \frac{{}^E d(T_{ij})}{dt}\mathbf{E}_j = \dot{T}_{ij}\mathbf{E}_j = \dot{T}_{ij} T_{kj} \mathbf{e}_k$$

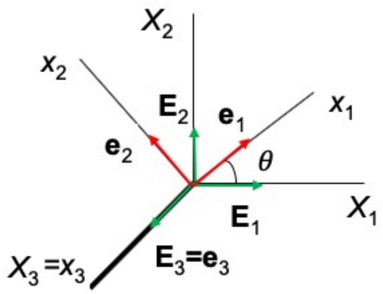
$\Rightarrow$

$$\frac{{}^E du_i}{dt} = \frac{{}^e du_i}{dt} + (\dot{T}_{ij} T_{kj}) u_k$$

$$\frac{{}^E d\mathbf{u}}{dt} = \frac{{}^e d\mathbf{u}}{dt} + \dot{\mathbf{T}}\mathbf{T}^T \mathbf{u}$$

Diferenciação de vetores: interpretação

$$\frac{{}^{E}d\mathbf{u}}{dt} = \frac{{}^{e}d\mathbf{u}}{dt} + \dot{\mathbf{T}}\mathbf{T}^{T}\,\mathbf{u}$$



$$T = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T^{T} = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dot{\mathbf{T}}\mathbf{T}^{T} = \begin{bmatrix} -\dot{\theta}s\theta & \dot{\theta}c\theta & 0 \\ -\dot{\theta}c\theta & -\dot{\theta}s\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \dot{\theta} & 0 \\ -\dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{Bmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \\ \dot{\mathbf{e}}_3 \end{Bmatrix} = \begin{bmatrix} 0 & \dot{\theta} & 0 \\ -\dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{Bmatrix} = \begin{Bmatrix} \dot{\theta}\mathbf{e}_2 \\ -\dot{\theta}\mathbf{e}_1 \\ 0 \end{Bmatrix}$$

Velocidade angular: $${}^{E}\boldsymbol{\omega}^{e} = \dot{\theta}\mathbf{e}_3 = \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix}$$

→

$$\dot{\mathbf{e}}_1 = \dot{\theta}\mathbf{e}_3 \times \mathbf{e}_1 = \dot{\theta}\mathbf{e}_2$$

$$\dot{\mathbf{e}}_2 = \dot{\theta}\mathbf{e}_3 \times \mathbf{e}_2 = -\dot{\theta}\mathbf{e}_1$$

$$\dot{\mathbf{e}}_3 = \dot{\theta}\mathbf{e}_3 \times \mathbf{e}_3 = 0$$

→

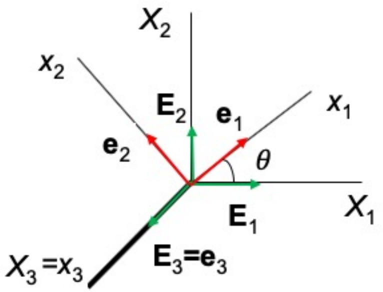
$$\dot{\mathbf{e}}_i = {}^{E}\boldsymbol{\omega}^{e} \times \mathbf{e}_i$$

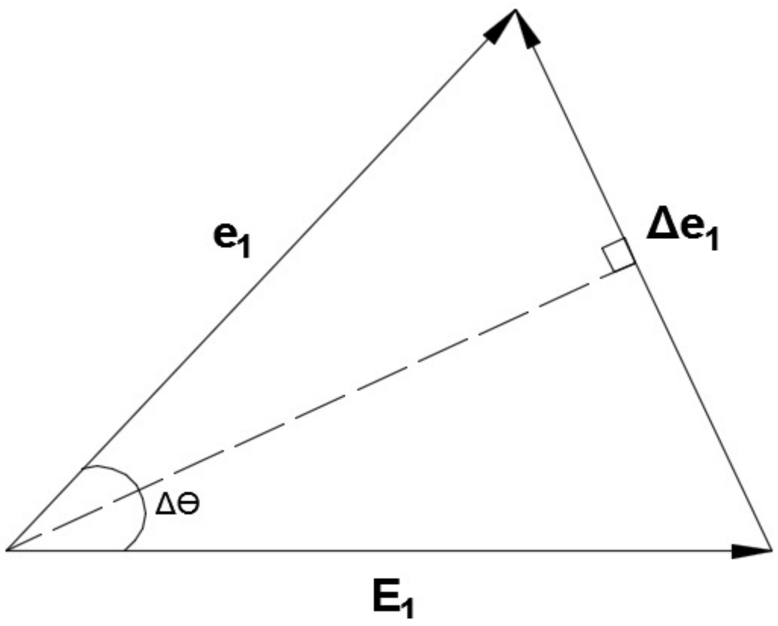
Portanto:

$$\frac{{}^{E}d\mathbf{u}}{dt} = \frac{{}^{e}d\mathbf{u}}{dt} + {}^{E}\boldsymbol{\omega}^{e} \times \mathbf{u}$$

Diferenciação de vetores: interpretação

$$\frac{{}^{E}d\mathbf{u}}{dt} = \frac{{}^{e}d\mathbf{u}}{dt} + {}^{E}\boldsymbol{\omega}^{e} \times \mathbf{u}$$





$$\dot{\mathbf{e}}_i = \frac{d\mathbf{e}_1}{dt} = \frac{\partial \mathbf{e}_1}{\partial \theta}\frac{\partial \theta}{\partial t} = \frac{\partial \mathbf{e}_1}{\partial \theta}\dot{\theta} = \lim_{\Delta\theta \to 0}\frac{\Delta \mathbf{e}_1}{\Delta \theta}\dot{\theta}$$

$$\Delta e_1 = |\Delta \mathbf{e}_1| = \Delta\theta$$

$\Delta \mathbf{e}_1$ - perpendicular a $\mathbf{E}_1$

Portanto: $\dot{\mathbf{e}}_1 = \dot{\theta}\mathbf{e}_2$