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Geometrically non-linear elastic model for a thin composite layer with wavy surfaces

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The geometrically non-linear elastic thin composite layer model is developed through the application of the modified asymptotic homogenization method. The set of local unit cell problems and the analytical formulae for the effective stiffness moduli of the non-linear homogenized plate accounting the higher order terms of the asymptotic expansions are derived. They make it possible to gain useful insight into the manner in which the geometrical and mechanical properties of the individual constituents affect the elastic properties of the composite layer with wavy surfaces. It is shown that in the limiting case of a homogeneous layer of constant thickness the derived asymptotic homogenization model converges to the geometrically non-linear mean-flexure plate theory. And the obtained expressions for the mid-surface strains converge to von Kármán's formulae. The derived non-linear homogenization model is illustrated by an example of a laminated plate.

1 Introduction

The preponderance of uses for composite materials is in the form of thin shells and plates, the optimum strength-to-weight characteristics of which offer engineers attractive alternatives for different applications. A large part of these applications is in the aerospace, structural and marine fields, where the composites are made of continuous fibers in polymeric matrix to obtain fiber-reinforced polymer laminated composite plates or shells. The geometry of such composite structures is governed by periodic configuration, i.e. reinforcements are regularly distributed with very significant coordinate effects, so as to reap the benefit of carrying smaller weights under certain loading conditions. However, the practical issue in the mechanics of advanced composites is the determination of the effective properties of these structures which will naturally be dependent on the spatial distribution of fibers, geometric characteristics, and the mechanical properties of the constituent materials involved.

The asymptotic analysis of thin-walled structures has been the focus of investigation for long time. The classical theory of shells is based on the assumption that normal to mid-plane before deformation remains straight and normal to the plane after deformation, and the effects of transverse shear strains were ignored, see, e.g., Novozhilov [1]. Consequently, the applicability of the general solution has been confined to plates strictly with limited thickness subjected to edge tractions through the use of a series of biharmonic functions [2]. In the Hencky-Mindlin theories, the displacements are expanded in powers of the thickness of the plate [3]. A geometrically non-linear theory associated with the classical plate theory was considered by Reissner [4]. The laminated composite plates are studied in Reddy [5].

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The continued interest in finding the solution for elastic problem for laminates made up of orthotropic layers has brought the attention to asymptotic expansion and its application into the analysis of composite plates with periodic structures. In the initial elastic problem for a composite plate with periodic structure, two small parameters, namely the plate thickness δ and an in-plane dimension ε of the periodicity cell, were considered. In the classical asymptotic homogenization theory, plates of constant thickness were considered with simultaneous reduction of all dimensions of the periodicity cells by Caillerie [6]. Homogeneous plates of rapidly varying thickness have been studied independently by Kohn and Vogelius [7–9] with the help of methods similar to those used by Caillerie [10]. Kalamkarov [11, 12] has generalized the above two approaches to the case of shells with rapidly varying material properties and thickness. In the works by Kalamkarov and Georgiades [13] and Hadjiloizi et al. [14], the general asymptotic homogenization model was expanded to the application of smart composite shells with periodically arranged actuators and varying thickness. Details on practically important applications of the micromechanical models based on application of the asymptotic homogenization method can be found in Kalamkarov and Kolpakov [15] and Kalamkarov et al. [16]. All these models have been developed in the framework of linear elasticity problem valid for the cases of small deformations of composite structures.

The present paper is aimed at developing higher order terms of the asymptotic expansions that model the deformations of thin composite layer with wavy surfaces in the framework of geometrically non-linear elasticity. To the ultimate objective, the modified asymptotic homogenization method is applied to the study of geometrically non-linear 3D elasticity problem without simplification of the Kirchhoff-Love hypothesis. Owing the small parameter $\varepsilon \sim \delta$, 3D problem is proved to be amenable to a rigorous asymptotic analysis unifying an asymptotic 3D to 2D process and the asymptotic homogenization of the composite material in tangential directions.

The paper is organized as follows. The problem is formulated in Sect. 2. The asymptotic homogenization analysis is conducted in Sect. 3, and the derived local unit-cell problems are solved in Sect. 4. The governing equations of the non-linear homogenized plate are formulated in Sect. 5. The comparison with the geometrically non-linear plate theory and the numerical example are given in Sects. 6 and 7.

2 Problem formulation

We apply the geometrically non-linear theory of elasticity to a thin 3D periodically inhomogeneous (composite) plane layer with wavy surfaces. And we introduce the dimensionless Cartesian coordinates x_1, x_2, x_3 such that the coordinate plane x_1, x_2 coincides with the mid-plane of layer (for $x_3 = 0$), while the x_3 axis is perpendicular.

Consider a 3D thin layer with wavy surfaces representing an inhomogeneous solid with the unit cell occupying domain Ω_δ, as shown in Fig. 1. The unit cell can be defined by the following inequalities:

$$-\frac{\delta h_1}{2} < x_1 < \frac{\delta h_1}{2}, \quad -\frac{\delta h_2}{2} < x_2 < \frac{\delta h_2}{2}, \quad x_3^- < x_3 < x_3^+, \tag{1}$$

where $x_3^{\pm} = \pm\frac{\delta}{2} \pm \delta F^{\pm}(\frac{x_1}{\delta h_1}, \frac{x_2}{\delta h_2})$.

Small parameter $\delta \ll 1$ determines the thickness of the unit cell; and h_1 and h_2 are the ratios of the corresponding tangential dimensions of the unit cell along x_1 and x_2 directions to the thickness of the unit cell.

The functions $F^{\pm}$ define the geometrical shape of top and bottom surfaces of the unit cell. They model shape of surface reinforcements. Functions $F^{\pm}$ are 1-periodic in corresponding variables $\frac{x_1}{\delta h_1}$ and $\frac{x_2}{\delta h_2}$. In particular case if there are no surface reinforcements, $F^{\pm} \equiv 0$.

As it is seen from the Eq. (1) the small thickness of the layer and both tangential dimensions of the unit cell are proportional to the small parameter δ, and therefore they have the same order of magnitude. Consequently the consideration of the present paper is aimed to the analysis of thin composite layers for which the periodic variations of material and geometrical inhomogeneities in tangential directions have the same order of smallness as the transversal thickness of the layer. This type of thin-walled composite structures is very common in numerous practical applications. Other possible types of composite structures would include two different limiting cases: first one with tangential scale of inhomogeneity much larger than thickness, and second one with tangential scale of inhomogeneity much smaller than thickness of composite layer. In the first case the composite layer can be treated as a 2D composite plate, and the standard 2D asymptotic homogenization can be applied, see e.g., [12, 15–17]. In the second case the composite layer is rather thick, and therefore a standard 3D asymptotic homogenization can be applied without consideration of a small thickness of layer, see e.g., [12, 15, 16]. The case analyzed in the present paper is more complicated because the asymptotic homogenization must be applied simultaneously with the asymptotic transition from 3D non-linear elasticity to 2D homogenized geometrically nonlinear plate. This case requires development and application of the modified asymptotic homogenization analysis.

The strains are related to the displacements through the non-linear relations

$$2e_{ij} = u_{i,j} + u_{j,i} + 2u_{k,i}u_{k,j}, \tag{2}$$

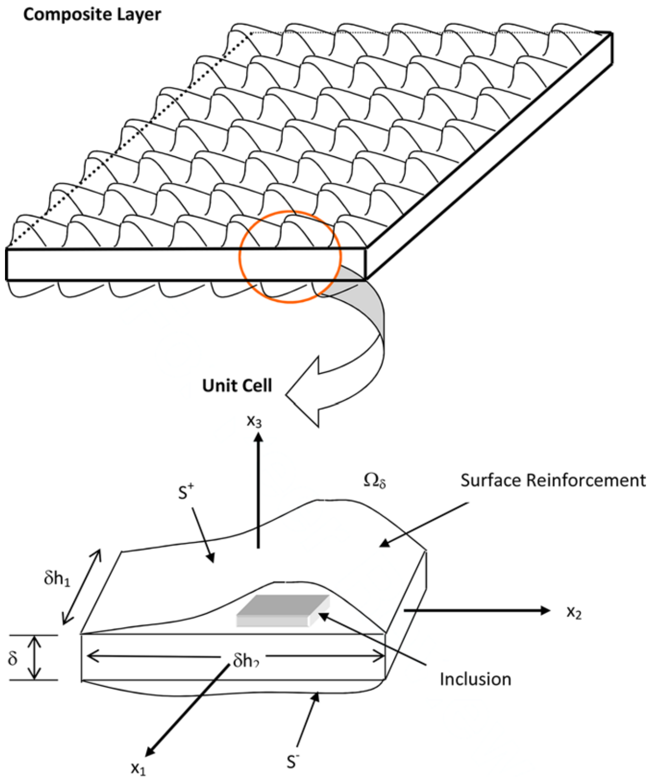

Fig. 1 Thin composite layer with wavy surfaces and its unit cell.

where $\frac{\partial u_j}{\partial x_i} = u_{j,i}$. Here and henceforth, all repeated indices are summed, and $i, j, k, l = 1, 2, 3$.

The non-linear equilibrium equations are written as follows:

$$t_{ij,i} + P_j = 0,$$
$$t_{ij} = \sigma_{ij} + \sigma_{li} u_{j,l}, \tag{3}$$

where P_j is a body force after the deformation.

In the geometrically non-linear elastic problem under study, the material is considered to be physically linearly elastic, and the physical components of stresses σ_{ij} and strains e_{kl} are connected by the Hooke's Law:

$$\sigma_{ij} = c_{ijkl} e_{kl}, \tag{4}$$

where c_{ijkl} are the elastic coefficients.

We assume that the surfaces of the layer are subjected to forces

$$t_{ij} n_j^{\pm} = \pm p_i^{\pm}, \tag{5}$$

where $n_j^{\pm}$ are the components of the unit normal vectors to the surfaces $S^{\pm}$, prior to deformation, and $p_i^{\pm}$ are the components of the surface forces acting in the already deformed layer.

3 Application of the asymptotic homogenization method

In the framework of the modified asymptotic homogenization method [11–16] we introduce the "fast" coordinates $y_1 = x_1/(\delta h_1)$, $y_2 = x_2/(\delta h_2)$, and $z = x_3/\delta$. The "fast" and "slow" variables (x_1, x_2, x_3) will be distinguished when performing differentiation. The solution of the problem is represented as an asymptotic series expansion in powers of the small parameter δ in the form (see [12, 15, 16]):

$$u_i = u_i^{(0)}(\vec{x}) + \delta u_i^{(1)}(\vec{x}, \vec{y}, z) + \delta^2 u_i^{(2)}(\vec{x}, \vec{y}, z) + \ldots, \tag{6}$$

where $\vec{x} = (x_1, x_2)$, $\vec{y} = (y_1, y_2)$ and the functions $u_i^{(l)}(\vec{x}, \vec{y}, z)$ for $l = 1, 2, \ldots$ are 1-periodic in fast variables y_1 and y_2.

Asymptotic behavior of the external forces is given as follows:

$$\begin{aligned} &P_v = \delta f_v(\vec{x}, \vec{y}, z), \quad P_3 = \delta^2 f_3(\vec{x}, \vec{y}, z), \\ &p_v^{\pm} = \delta^2 g_v^{\pm}(\vec{x}, \vec{y}), \quad p_3^{\pm} = \delta^3 g_3^{\pm}(\vec{x}, \vec{y}), \quad \nu = 1, 2, \end{aligned} \tag{7}$$

where all above functions are periodic in y_1 and y_2, with the unit cell Ω defined by:

$$y_1, y_2 \in (-0.5, 0.5), z \in (z^-, z^+), \quad \text{where} \quad z^{\pm} = \pm 0.5 \pm F^{\pm}(\vec{y}). \tag{8}$$

Likewise, the periodicity property is applied in the elastic coefficients $c_{ijkl}(\vec{y}, z)$, which are visualized as piecewise-smooth functions undergoing discontinuities of the first-kind at the non-intersecting interfaces between the dissimilar constituents of the composite material.

It then follows from Eqs. (3)–(6) that

$$\begin{aligned} &\sigma_{ij} = \sigma_{ij}^{(0)} + \delta\sigma_{ij}^{(1)} + \delta^2\sigma_{ij}^{(2)} + \cdots, \\ &t_{ij} = t_{ij}^{(0)} + \delta t_{ij}^{(1)} + \delta^2 t_{ij}^{(2)} + \cdots. \end{aligned} \tag{9}$$

Using the Eq. (7), (9) in Eq. (3) yields:

$$\begin{aligned} &\delta^{-1} H_j^{(-1)} + H_j^{(0)} + \delta H_j^{(1)} + \delta^2 H_j^{(2)} + \cdots = 0, \\ &H_j^{(-1)} = t_{3j,3}^{(0)} + \frac{1}{h_\alpha} t_{\alpha j,\alpha}^{(0)}, \\ &H_j^{(0)} = t_{\alpha j,\alpha}^{(0)} + t_{3j,3}^{(1)} + \frac{1}{h_\alpha} t_{\alpha j,\alpha}^{(1)}, \\ &H_j^{(1)} = t_{\alpha j,\alpha}^{(1)} + t_{3j,3}^{(2)} + \frac{1}{h_\alpha} t_{\alpha j,\alpha}^{(2)} + f_j(\delta_{j1} + \delta_{j2}), \\ &H_j^{(2)} = t_{\alpha j,\alpha}^{(2)} + t_{3j,3}^{(3)} + \tfrac{1}{h_\alpha} t_{\alpha j,\alpha}^{(3)} + f_j\delta_{j3}, \end{aligned} \tag{10}$$

$$\left(t_{ij}^{(0)} + \delta t_{ij}^{(1)} + \delta^2 t_{ij}^{(2)} + \delta^3 t_{ij}^{(3)} + \cdots\right) n_i^{\pm} = \delta^2 g_j^{\pm}(\delta_{j1} + \delta_{j2}) \pm \delta^3 g_j^{\pm}\delta_{j3}, \quad (z = z^{\pm}), \tag{11}$$

where δ_{jl} is the Kronecker delta; the Greek indices (i.e., α, β, λ, μ) take values 1 and 2, while Latin indices (i.e., i, j, l, n, m) take values 1, 2, 3.

We introduce the following differential operator:

$$\boldsymbol{L}_{ijn} = c_{ijn\mu} \frac{1}{h_\mu} \frac{\partial}{\partial y_\mu} + c_{ijn3} \frac{\partial}{\partial z}. \tag{12}$$

The leading terms in Eq. (9) may be then written as

$$\begin{aligned} &\sigma_{ij}^{(0)} = \boldsymbol{L}_{ijk} u_k^{(1)} + c_{ijk\alpha} u_{k,\alpha}^{(0)} + \frac{1}{2h_\mu} u_{m|\mu}^{(1)} \boldsymbol{L}_{ij\mu} u_m^{(1)} + \frac{1}{2} u_{m|3}^{(1)} \boldsymbol{L}_{ij3} u_m^{(1)} + u_{m,\alpha}^{(0)} \boldsymbol{L}_{ij\alpha} u_m^{(1)} + \frac{1}{2} c_{ij\alpha\beta} u_{m,\alpha}^{(0)} u_{m,\beta}^{(0)}, \\ &t_{ij}^{(0)} = \sigma_{ij}^{(0)} + \sigma_{i\beta}^{(0)} u_{j,\beta}^{(0)} + \sigma_{i3}^{(0)} u_{j|3}^{(1)} + \frac{1}{h_\beta} \sigma_{i\beta}^{(0)} u_{j|\beta}^{(1)}, \end{aligned} \tag{13}$$

where the derivatives with respect to "fast" variables are denoted $\frac{\partial u_j}{\partial y_l} = u_{j|l}$.

The problem of determining $t_{ij}^{(0)}$ follows from Eqs. (10), (11) as

$$H_j^{(-1)} = 0, \quad t_{ij}^{(0)} n_j^{\pm} = 0, \quad \text{at } z = z^{\pm}. \tag{14}$$

The substitution of Eq. (13) yields a problem for the functions $u_k^{(1)}$, in which we shall ignore the terms containing products of three or more derivatives of displacement components with respect to the "slow" coordinates x_1, x_2.

The solution of the problem in Eqs. (13) and (14) can be represented in the form:

$$u_k^{(1)} = v_k^{(1)}(\vec{x}) + U_k^{n\mu}(\vec{y}, z)\, u_{n,\mu}^{(0)} + W_k^{mn\lambda\mu}(\vec{y}, z)\, u_{m,\mu}^{(0)} u_{n,\mu}^{(0)} \tag{15}$$

with the provision that the functions $U_k^{n\mu}(\vec{y}, z)$ and $W_k^{mn\lambda\mu}(\vec{y}, z)$ are 1-periodic in y_1 and y_2 and solve the following local unit-cell problems:

$$\begin{aligned} &\frac{1}{h_\beta} b_{i\beta|\beta}^{n\mu} + b_{i3|3}^{n\mu} = 0, \quad b_{ij}^{n\mu} = \boldsymbol{L}_{ijk} U_k^{n\mu} + c_{ijn\mu}, \\ &b_{ij}^{n\mu} n_j^{\pm} = 0 \quad \left(z = z^{\pm}\right), \end{aligned} \tag{16}$$

$$\begin{aligned} &\frac{1}{h_\beta}\left(B_{i\beta}^{mn\lambda\mu} + b_{\alpha\beta}^{m\lambda}\frac{1}{h_\alpha}U_{i|\alpha}^{n\mu} + b_{3\beta}^{m\lambda}U_{i|3}^{n\mu}\right)_{|\beta} + \left(B_{i3}^{mn\lambda\mu} + b_{\alpha 3}^{m\lambda}\frac{1}{h_\alpha}U_{i|\alpha}^{n\mu} + b_{33}^{m\lambda}U_{i|3}^{n\mu}\right)_{|3} = 0, \\ &B_{ij}^{mn\lambda\mu} n_j^{\pm} = 0 \quad (z = z^{\pm}), \end{aligned} \tag{17}$$

$$B_{ij}^{mn\lambda\mu} = \boldsymbol{L}_{ijk} W_k^{mn\lambda\mu} + \frac{1}{2h_\alpha} U_{k|\alpha}^{m\lambda} \boldsymbol{L}_{ij\alpha} U_k^{n\mu} + \frac{1}{2} U_{k|3}^{m\lambda} \boldsymbol{L}_{ij3} U_k^{n\mu} + \boldsymbol{L}_{ij\lambda} U_m^{n\mu} + \frac{1}{2} c_{ij\lambda\mu} \delta_{mn}. \tag{18}$$

It is seen that problems given by Eq. (16) coincide with local problems for a thin plate in the framework of linear elasticity theory, see [12, 15, 16]. Local unit-cell problems given by Eqs. (17) and (18) are principally new, and they are responsible for non-linear problem studied in the present paper.

Note that at the interfaces where discontinuities in material properties of composite occur, the additional continuity conditions must be added to the above local problems given by Eqs. (16)–(18). These continuity conditions will be discussed later.

4 Solution of the local unit-cell problems

It can be shown that local problem given by the Eq. (16) has an exact solution for $n = 3$ and $\mu = 1$ or 2, given by the functions

$$U_1^{31} = -z, \quad U_2^{31} = U_3^{31} = 0, \quad U_2^{32} = -z, \quad U_1^{32} = U_3^{32} = 0 \tag{19}$$

and as a result (for $\mu = 1$ and 2),

$$b_{ij}^{3\mu} = 0. \tag{20}$$

Substituting above relations into the Eq. (17) reduces it to a much simpler form for $mn = 33$, as follows:

$$\begin{aligned} &\frac{1}{h_\beta} B_{i\beta|\beta}^{33\lambda\mu} + B_{i3|3}^{33\lambda\mu} = 0, \\ &B_{ij}^{33\lambda\mu} n_j^{\pm} = 0 \quad (z = z^{\pm}), \\ &B_{ij}^{33\lambda\mu} = \boldsymbol{L}_{ijk} W_k^{33\lambda\mu} + \frac{1}{2} c_{ij33} \delta_{\lambda\mu} + \frac{1}{2} c_{ij\lambda\mu}. \end{aligned} \tag{21}$$

Comparing the local problems given by Eq. (16) for the functions $U_k^{\lambda\mu}$ and Eq. (21) for $W_k^{33\lambda\mu}$ it can be shown that

$$\begin{aligned} &W_\alpha^{33\lambda\mu} = \frac{1}{2} U_\alpha^{\lambda\mu} \quad (\alpha = 1, 2), \\ &W_3^{33\lambda\mu} = \frac{1}{2}\left(U_3^{\lambda\mu} - z\delta_{\lambda\mu}\right), \end{aligned} \tag{22}$$

and using this in Eq. (21) for $B_{ij}^{33\lambda\mu}$ yields

$$B_{ij}^{33\lambda\mu} = \frac{1}{2} b_{ij}^{\lambda\mu} \tag{23}$$

after comparing with Eq. (16) for $b_{ij}^{\lambda\mu}$.

In what follows, we substitute Eq. (15) into the Eq. (13) and use the notation of Eqs. (16), (17) to obtain:

$$\sigma_{ij}^{(0)} = b_{ij}^{\lambda\mu} u_{\lambda,\mu}^{(0)} + B_{ij}^{mn\lambda\mu} u_{m,\lambda}^{(0)} u_{n,\mu}^{(0)}. \tag{24}$$

Application of the asymptotic homogenization technique and the use of Eq. (14) now yields the problem from which the leading order terms in Eqs. (10) and (14) can be determined:

$$H_j^{(0)} = \left\langle H_j^{(0)} \right\rangle, \quad t_{ij}^{(1)} n_i^{\pm} = 0 \quad (z = z^{\pm}), \tag{25}$$

where the volume average is defined as follows:

$$\langle \varphi \rangle = \frac{1}{|\Omega|} \int\limits_{\Omega} \varphi \, dy_1 \, dy_2 \, dz. \tag{26}$$

Using the periodicity in y_1 and y_2 and conditions (25) at $z = z^{\pm}$ it is found from Eq. (13) that

$$\left\langle H_j^{(0)} \right\rangle = t_{\alpha j,\alpha}^{(0)}. \tag{27}$$

Following the earlier found solution of the linear version of the problem (25) and (27), we write

$$u_1^{(0)} = u_2^{(0)} = 0, \quad u_3^{(0)} = w(\vec{x}), \quad v_3^{(1)}(\vec{x}) = 0. \tag{28}$$

This yields

$$\begin{aligned} u_\alpha^{(1)} &= v_\alpha^{(1)}(\vec{x}) - z w_{,\alpha} + \frac{1}{2} U_\alpha^{\lambda\mu} w_{,\lambda} w_{,\mu}, \\ u_3^{(1)} &= \frac{1}{2} \left(U_3^{\lambda\mu} - z \delta_{\lambda\mu} \right) w_{,\lambda} w_{,\mu}. \end{aligned} \tag{29}$$

The Eqs. (24) and (23) yield on account of Eqs. (15) and (22):

$$\sigma_{ij}^{(0)} = \frac{1}{2} b_{ij}^{\lambda\mu} w_{,\lambda} w_{,\mu}. \tag{30}$$

Using Eqs. (2)–(4), (28), and (29) we find

$$\begin{aligned} \sigma_{ij}^{(1)} &= \boldsymbol{L}_{ijk} u_k^{(2)} + c_{ij\alpha\beta} \varepsilon_{\alpha\beta}^{(1)} + z c_{ij\alpha\beta} \tau_{\alpha\beta} + w_{,\alpha} \left(\boldsymbol{L}_{ij\alpha} u_3^{(2)} - \boldsymbol{L}_{ij3} u_\alpha^{(2)} \right) \\ &\quad - c_{ij3\beta} w_{,\alpha} \varepsilon_{\alpha\beta}^{(1)} - c_{ijm\beta} U_m^{\alpha\mu} w_{,\alpha} \tau_{\mu\beta}, \\ t_{ij}^{(1)} &= \sigma_{ij}^{(1)} + \sigma_{i\beta}^{(1)} w_{,\beta} \delta_{j3} - \sigma_{i3}^{(1)} w_{,\beta} \delta_{j\beta}, \end{aligned} \tag{31}$$

denoting,

$$\varepsilon_{\alpha\beta}^{(1)} = v_{\alpha,\beta}^{(1)}, \quad \tau_{\alpha\beta} = -w_{,\alpha\beta}, \tag{32}$$

where in accordance with the earlier notation, $w_{,\alpha\beta} = \frac{\partial^2 w}{\partial x_\alpha \partial x_\beta}$.

Now if we substitute Eqs. (27) and (31), (32) into Eq. (25) and make use of Eqs. (13) and (30), a problem for determining the functions $u_k^{(2)}$ will be obtained, the solution of which may be represented in the following form:

$$u_k^{(2)} = U_k^{\lambda\mu} \varepsilon_{\lambda\mu}^{(1)} + V_k^{\lambda\mu} \tau_{\lambda\mu} + Q_k^{\alpha\lambda\mu} w_{,\alpha} \varepsilon_{\lambda\mu}^{(1)} + R_k^{\alpha\lambda\mu} w_{,\alpha} \tau_{\lambda\mu}. \tag{33}$$

Here the functions $U_k^{\lambda\mu}(\vec{y}, z)$, $V_k^{\lambda\mu}(\vec{y}, z)$, $Q_k^{\alpha\lambda\mu}(\vec{y}, z)$, and $R_k^{\alpha\lambda\mu}(\vec{y}, z)$ are 1-periodic in y_1 and y_2 with the unit cell Ω and solve the following local unit cell problems:

$$\frac{1}{h_\beta} b_{i\beta|\beta}^{\lambda\mu} + b_{i3|3}^{\lambda\mu} = 0, \quad b_{ij}^{\lambda\mu} n_j^{\pm} = 0, \quad \text{at } z = z^{\pm}, \quad \left(b_{ij}^{\lambda\mu} \leftrightarrow b_{ij}^{*\lambda\mu} \leftrightarrow q_{ij}^{\alpha\lambda\mu} \right), \tag{34}$$

$$\frac{1}{h_\beta} r_{i\beta|\beta}^{\alpha\lambda\mu} + r_{i3|3}^{\alpha\lambda\mu} = b_{i\mu}^{\lambda\alpha} - b_{i\mu}^{\lambda\alpha}, \quad r_{ij}^{\alpha\lambda\mu} n_j^{\pm} = 0, \quad \text{at } z = z^{\pm}, \tag{35}$$

$$\begin{aligned} b_{ij}^{\lambda\mu} &= \boldsymbol{L}_{ijk} U_k^{\lambda\mu} + c_{ij\lambda\mu}, \quad b_{ij}^{*\lambda\mu} = \boldsymbol{L}_{ijk} V_k^{\lambda\mu} + z c_{ij\lambda\mu}, \\ q_{ij}^{\alpha\lambda\mu} &= \boldsymbol{L}_{ijk} Q_k^{\alpha\lambda\mu} + \boldsymbol{L}_{ij\alpha} U_3^{\lambda\mu} - \boldsymbol{L}_{ij3} U_\alpha^{\lambda\mu} - c_{ij3\mu} \delta_{\alpha\lambda}, \\ r_{ij}^{\alpha\lambda\mu} &= \boldsymbol{L}_{ijk} R_k^{\alpha\lambda\mu} + \boldsymbol{L}_{ij\alpha} V_3^{\lambda\mu} - \boldsymbol{L}_{ij3} V_\alpha^{\lambda\mu} - c_{ijm\mu} U_m^{\alpha\lambda}, \end{aligned} \tag{36}$$

where sign $\leftrightarrow$ denotes replacement of the corresponding functions.

As it was mentioned earlier, the continuity conditions on interfaces where discontinuities in material properties of composite occur must be added to the foregoing equations. In the case of perfect contact of different constituents of composite material (e.g., inclusion and matrix), the continuity conditions are given in the following form:

$$[\![U_k^{\lambda\mu} = 0]\!] \quad \left(U_k^{\lambda\mu} \leftrightarrow V_k^{\lambda\mu} \leftrightarrow Q_k^{\alpha\lambda\mu} \leftrightarrow R_k^{\alpha\lambda\mu} \right),$$

$$\left[\!\left[\frac{1}{h_\beta} n_\beta^{(k)} b_{i\beta}^{\lambda\mu} + n_3^{(k)} b_{i3}^{\lambda\mu} = 0\right]\!\right] \quad \left(b_{ij}^{\lambda\mu} \leftrightarrow b_{ij}^{*\lambda\mu} \leftrightarrow q_{ij}^{\alpha\lambda\mu} \leftrightarrow r_{ij}^{\alpha\lambda\mu}\right), \tag{37}$$

where $[\![\cdots]\!]$ denotes a jump of function over the interface, and $n_i^{(k)}$ denotes the unit normal vector to the interface, related to the coordinate system y_1, y_2, z. This is in contrast to the $z = z^\pm$ conditions in the local problems (16)–(18), (21), and (34)–(36), where $n_i^\pm$, the unit normal to the surfaces $S^\pm$, are related to the coordinate system x_1, x_2, x_3. In actually solving these problems, however, it proves more convenient to rewrite them using the unit normal vectors $n_i^{\pm(y)}$ related to y_1, y_2, z.

The local problems given by the Eqs. (34)–(37) are linear in the unknown functions, and their solutions are unique up to constant terms. This ambiguity is removed by the conditions

$$\left\langle U_k^{\lambda\mu}\right\rangle_y = 0 \quad when \quad z = 0 \quad \left(U_k^{\lambda\mu} \leftrightarrow V_k^{\lambda\mu} \leftrightarrow Q_k^{\alpha\lambda\mu} \leftrightarrow R_k^{\alpha\lambda\mu}\right), \tag{38}$$

where $\langle \ldots_y \rangle$ indicates average with respect to y_1 and y_2 only.

Note that the problems for the functions $U_k^{\lambda\mu}$ and $V_k^{\lambda\mu}$ are identical to the corresponding local problems obtained in the framework of the linear theory of elasticity, see [12, 15], and these functions are considered to be known in the remaining two problems for the functions $Q_k^{\alpha\lambda\mu}$ and $R_k^{\alpha\lambda\mu}$.

Substituting Eq. (33) into Eq. (31) and using the notation of Eq. (36) we arrive at

$$\sigma_{ij}^{(1)} = b_{ij}^{\lambda\mu} \varepsilon_{\lambda\mu}^{(1)} + b_{ij}^{*\lambda\mu} \tau_{\lambda\mu} + q_{ij}^{\alpha\lambda\mu} w_{,\alpha} \varepsilon_{\lambda\mu}^{(1)} + r_{ij}^{\alpha\lambda\mu} w_{,\alpha} \tau_{\lambda\mu}. \tag{39}$$

Returning now to the Eqs. (34) and (35), we average their left-hand sides after first multiplying them by z and z^2 and we take into account, in doing so, the $z = z^\pm$ boundary conditions and periodicity in y_1 and y_2. This yields the following relations for the effective elastic moduli of the homogenized layer:

$$\begin{aligned} &\left\langle b_{i3}^{\lambda\mu}\right\rangle = \left\langle z b_{i3}^{\lambda\mu}\right\rangle = 0, \quad b_{i3}^{\lambda\mu} \leftrightarrow b_{i3}^{*\lambda\mu} \leftrightarrow q_{i3}^{\alpha\lambda\mu}, \\ &\left\langle r_{i3}^{\alpha\lambda\mu}\right\rangle = \langle z\rangle \left\langle b_{i\mu}^{\alpha\lambda}\right\rangle - \left\langle z b_{i\mu}^{\alpha\lambda}\right\rangle. \end{aligned} \tag{40}$$

The following symmetry properties hold:

$$b_{ij}^{mn} = b_{ji}^{mn} = b_{ij}^{nm} \quad \left(b_{ij}^{mn} \leftrightarrow b_{ij}^{*mn}\right). \tag{41}$$

Finally, from the Eqs. (13) and (31):

$$\begin{aligned} &\left\langle t_{\alpha\beta}^{(0)}\right\rangle = \left\langle \sigma_{\alpha\beta}^{(0)}\right\rangle, \quad \left\langle t_{\alpha3}^{(0)}\right\rangle = \left\langle \sigma_{\alpha\beta}^{(0)}\right\rangle w_{,\beta}, \\ &\left\langle t_{\alpha\beta}^{(1)}\right\rangle = \left\langle \sigma_{\alpha\beta}^{(1)}\right\rangle, \quad \left\langle z t_{\alpha\beta}^{(1)}\right\rangle = \left\langle z \sigma_{\alpha\beta}^{(1)}\right\rangle, \\ &\left\langle t_{\alpha3}^{(1)}\right\rangle = \left\langle \sigma_{\alpha\beta}^{(1)}\right\rangle w_{,\beta} + \left\langle r_{\alpha3}^{\beta\lambda\mu}\right\rangle w_{,\beta} \tau_{\lambda\mu}, \end{aligned} \tag{42}$$

where Eqs. (28)–(30), (39), and (40) have been also used.

5 Governing equations of the non-linear homogenized plate

The problem addressed next is to derive a system of equations for $v_1^{(1)}(\vec{x})$, $v_1^{(1)}(\vec{x})$, and $w(\vec{x})$, functions of the "slow" variables entering the Eqs. (30), (32), (39); as well as the displacement vector given in Eqs. (6), (28), (29), (33). We begin by writing the following terms in Eq. (10):

$$\begin{aligned} &\left\langle H_\beta^{(0)}\right\rangle + \delta H_\beta^{(1)} = 0 \quad (\beta = 1, 2), \\ &\left\langle H_3^{(0)}\right\rangle + \delta H_3^{(1)} + \delta^2 H_3^{(2)} = 0, \end{aligned} \tag{43}$$

where Eqs. (14) and (25) were used. Applying the averaging operator (26), and using the conditions in (11), relations (10) and (27) and the periodicity in y_1 and y_2, we find that

$$\begin{aligned} &\left\langle t_{\alpha\beta}^{(0)}\right\rangle_{,\alpha} + \delta\left(\left\langle t_{\alpha\beta}^{(1)}\right\rangle_{,\alpha} + g_\beta + \langle f_\beta\rangle\right) = 0, \\ &\left\langle t_{\alpha3}^{(0)}\right\rangle_{,\alpha} + \delta\left\langle t_{\alpha3}^{(1)}\right\rangle_{,\alpha} + \delta^2\left(\left\langle t_{\alpha3}^{(2)}\right\rangle_{,\alpha} + g_3 + \langle f_3\rangle\right) = 0, \end{aligned} \tag{44}$$

where

$$g_j = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(g_j^+ \omega^+ + g_j^+ \omega^- \right) dy_1 \, dy_2,$$

and where

$$\omega^\pm = \left[1 + \frac{1}{h_1^2} \left(\frac{\partial F^\pm}{\partial y_1} \right)^2 + \frac{1}{h_2^2} \left(\frac{\partial F^\pm}{\partial y_2} \right)^2 \right]^{\frac{1}{2}}$$

in the Cartesian coordinates.

From the first of Eq. (43), multiplying by z and averaging, we obtain:

$$\langle z \rangle \left\langle t_{\alpha\beta,\alpha}^{(0)} \right\rangle + \delta \left(\left\langle z t_{\alpha\beta,\alpha}^{(1)} \right\rangle - \left\langle t_{3\beta}^{(2)} \right\rangle + m_\beta + \langle z f_\beta \rangle \right) = 0, \tag{45}$$

where

$$m_\beta = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(z^+ g_\beta^+ \omega^+ + z^- g_\beta^- \omega^- \right) dy_1 \, dy_2.$$

Note that the functions $g_j(\vec{x})$ and $m_\beta(\vec{x})$, associated with external loads, coincide in a linear approximation with the functions $r_\lambda(\vec{x})$, $q_3(\vec{x})$ and $\rho_\mu(\vec{x})$ defined as follows:

$$r_\lambda(\vec{x}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\omega^+ r_\lambda^+ + \omega^- r_\lambda^- \right) dy_1 \, dy_2,$$

$$q_3(\vec{x}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\omega^+ q_3^+ + \omega^- q_3^- \right) dy_1 \, dy_2,$$

$$\rho_\mu(\vec{x}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(z^+ \omega^+ r_\mu^+ + z^- \omega^- r_\mu^- \right) dy_1 \, dy_2.$$

Eliminating $\langle t_{\alpha 3}^{(2)} \rangle_{,\alpha}$ from the second of Eq. (44) and from Eq. (45), as well as noting that $\langle t_{3\alpha}^{(2)} \rangle = \langle t_{\alpha 3}^{(2)} \rangle$ within the accuracy of the calculation, we find:

$$\left\langle t_{\alpha 3}^{(0)} \right\rangle_{,\alpha} + \delta \left\langle t_{\alpha 3}^{(1)} \right\rangle_{,\alpha} + \delta \left[\langle z \rangle \left\langle t_{\alpha\beta}^{(0)} \right\rangle_{,\alpha} + \delta \left(\left\langle z t_{\alpha\beta}^{(1)} \right\rangle_{,\alpha} + m_\beta + \langle z f_\beta \rangle \right) \right]_{,\beta} + \delta^2 \left(g_3 + \langle f \rangle \right) = 0. \tag{46}$$

Now, from Eqs. (30), (32), (40), and (42) it follows that

$$\left\langle t_{\alpha 3}^{(1)} \right\rangle + \langle z \rangle \left\langle t_{\alpha\beta}^{(0)} \right\rangle_{,\beta} = \left\langle \sigma_{\alpha\beta}^{(1)} \right\rangle w_{,\beta} + \left\langle z \sigma_{\alpha\beta}^{(0)} \right\rangle_{,\beta},$$

which, combined with (42) reduces (46) to:

$$\left[\left(\left\langle \sigma_{\alpha\beta}^{(0)} \right\rangle + \delta \left\langle \sigma_{\alpha\beta}^{(1)} \right\rangle \right) w_{,\beta} \right]_{,\alpha} + \delta \left[\left(\left\langle z \sigma_{\alpha\beta}^{(0)} \right\rangle + \delta \left\langle z \sigma_{\alpha\beta}^{(1)} \right\rangle \right)_{,\alpha} + \delta \left(m_\beta + \langle z f_\beta \rangle \right) \right]_{,\beta} + \delta^2 \left(g_3 + \langle f_3 \rangle \right) = 0. \tag{47}$$

Using Eq. (42), the first of Eq. (44) can be put into the form:

$$\left(\left\langle \sigma_{\alpha\beta}^{(0)} \right\rangle + \delta \left\langle \sigma_{\alpha\beta}^{(1)} \right\rangle \right)_{,\alpha} + \delta \left(g_\beta + \langle f_\beta \rangle \right) = 0 \quad (\beta = 1, 2). \tag{48}$$

The system of Eqs. (47), (48) must be complemented by the elastic relations of the homogenized plate. We introduce

$$v_\lambda(\vec{x}) = \delta v_\lambda^{(1)}(\vec{x}), \quad \varepsilon_{\lambda\mu} = \delta \varepsilon_{\lambda\mu}^{(1)} = v_{\lambda,\mu} \tag{49}$$

in accordance with Eq. (32), and averaging Eqs. (30) and (39), we obtain the elastic relations of the form:

$$\left\langle z^l\sigma^{(0)}_{\alpha\beta}\right\rangle+\delta\left\langle z^l\sigma^{(1)}_{\alpha\beta}\right\rangle=\left\langle z^l b^{\lambda\mu}_{\alpha\beta}\right\rangle\left(\varepsilon_{\lambda\mu}+\frac{1}{2}w_{,\lambda}w_{,\mu}\right)+\delta\left\langle z^l b^{*\lambda\mu}_{\alpha\beta}\right\rangle\tau_{\lambda\mu}+\left\langle z^l q^{\theta\lambda\mu}_{\alpha\beta}\right\rangle w_{,\theta}\varepsilon_{\lambda\mu}$$
$$+\delta\left\langle z^l r^{\theta\lambda\mu}_{\alpha\beta}\right\rangle w_{,\theta}\tau_{\lambda\mu}\quad(l=0,1)\,. \tag{50}$$

This, when substituted into Eqs. (47), (48) yields the desired system of three governing equations for the functions $v_1(\vec{x})$, $v_2(\vec{x})$, and $w(\vec{x})$. Together with the solutions of the local problems given by the Eqs. (34)–(38), these functions enable us to calculate very accurately the displacement vector components which, using Eqs. (6), (28), (29), and (33) along with Eq. (49), are found to be given by:

$$\begin{aligned}u_\alpha&=v_\alpha\left(\vec{x}\right)-\gamma w_{,\alpha}+\delta U^{\lambda\mu}_\alpha\left(\varepsilon_{\lambda\mu}+\frac{1}{2}w_{,\lambda}w_{,\mu}\right)+\delta Q^{\beta\lambda\mu}_\alpha w_{,\beta}\varepsilon_{\lambda\mu}+\delta^2V^{\lambda\mu}_\alpha\tau_{\lambda\mu}\\&\quad+\delta^2R^{\beta\lambda\mu}_\alpha w_{,\beta}\tau_{\lambda\mu}+\cdots\quad(\alpha=1,2)\,,\\u_3&=w\left(\vec{x}\right)-\frac{\gamma}{2}w_{,\beta}w_{,\beta}+\delta U^{\lambda\mu}_3\left(\varepsilon_{\lambda\mu}+\frac{1}{2}w_{,\lambda}w_{,\mu}\right)+\delta Q^{\beta\lambda\mu}_3 w_{,\beta}\varepsilon_{\lambda\mu}+\delta^2V^{\lambda\mu}_3\tau_{\lambda\mu}\\&\quad+\delta^2R^{\beta\lambda\mu}_3 w_{,\beta}\tau_{\lambda\mu}+\cdots\,.\end{aligned}\tag{51}$$

It will be recognized that Eqs. (50) and (51), derived in the framework of the geometrically non-linear theory of elasticity, generalize the corresponding linear results.

6 Comparison of homogenization model and geometrically non-linear plate theory

Let us now discuss how the above asymptotic homogenization model relates to the geometrically non-linear theory of plates. We begin by writing down, with reference to Eq. (9), the leading order expressions for the stress and moment resultants:

$$\begin{aligned}N_{\alpha\beta}&=\delta\left\langle\sigma^{(0)}_{\alpha\beta}\right\rangle+\delta^2\left\langle\sigma^{(1)}_{\alpha\beta}\right\rangle,\\M_{\alpha\beta}&=\delta^2\left\langle z\sigma^{(0)}_{\alpha\beta}\right\rangle+\delta^3\left\langle z\sigma^{(1)}_{\alpha\beta}\right\rangle.\end{aligned}\tag{52}$$

Using Eq. (52) and Eq. (7) in Eqs. (47) and (48), it is seen that these latter coincide with the equations of equilibrium written in terms of projections onto the non-deformed axes in the framework of the non-linear mean-flexure plate theory as discussed, for example, in [1]. Evidently, both material inhomogeneity and thickness have their effect on the values of the effective elastic moduli that appear as coefficients in the elastic relations (50).

In the limiting case of a homogeneous isotropic layer of constant thickness, in which $c_{ijkl}=\text{const}$, $F^{\pm}\equiv 0$, the elastic coefficients of interest are given by:

$$c_{ijmn}=\frac{E}{2\left(1+\nu\right)}\left(\frac{2\nu}{1-2\nu}\delta_{ij}\delta_{mn}+\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}\right). \tag{53}$$

The coordinates y_1 and y_2 do not in fact play a role, and the local problems given by the Eqs. (33)–(38) have exact analytical solutions. As far as the functions $U^{\lambda\mu}_k$ and $V^{\lambda\mu}_k$ are concerned, the non-zero solutions of the local problems and the corresponding effective moduli have already been found as follows (see [12, 15]):

$$U^{11}_3=U^{22}_3=-\frac{\nu z}{1-\nu},\quad V^{11}_3=V^{22}_3=\frac{\nu z^2}{2\left(1-\nu\right)},$$

$$\left\langle b^{11}_{11}\right\rangle=\left\langle b^{22}_{22}\right\rangle=\frac{E}{1-\nu^2},\quad\left\langle b^{22}_{11}\right\rangle=\left\langle b^{11}_{22}\right\rangle=\frac{\nu E}{1-\nu^2},\quad\left\langle b^{12}_{12}\right\rangle=\frac{E}{2\left(1+\nu\right)},\quad\left\langle zb^{\beta\lambda}_{\mu\theta}\right\rangle=0,\quad\left\langle zb^{\beta\lambda}_{\mu\theta}\right\rangle=\frac{1}{12}\left\langle b^{\beta\lambda}_{\mu\theta}\right\rangle. \tag{54}$$

As for the functions $Q_k^{\alpha\lambda\mu}$ and $R_k^{\alpha\lambda\mu}$, the set of non-zero solutions of the pertinent local problems are as follows:

$$Q_1^{111} = Q_2^{222} = \frac{z}{1-\nu}, \quad Q_1^{122} = Q_2^{211} = \frac{\nu z}{1-\nu},$$
$$Q_1^{221} = Q_2^{112} = z, \quad R_1^{122} = Q_2^{211} = \frac{\nu z^2}{2(1-\nu)}, \quad R_1^{221} = Q_2^{112} = -\frac{\nu z^2}{2(1-\nu)}. \tag{55}$$

Substituting this into Eq. (36) we obtain:

$$q_{\alpha\beta}^{\theta\lambda\mu} = 0, \quad r_{\alpha\beta}^{\theta\lambda\mu} = 0. \tag{56}$$

If we substitute Eq. (54) and Eq. (56) into Eqs. (50) and (52), we arrive at elastic relations of the form usually adopted in geometrically non-linear mean-flexure plate theory. For the mid-surface strains, we have, by use of Eqs. (52) and (49),

$$\varepsilon_1 = v_{1,1} + \frac{1}{2}w_{,1}^2, \quad \varepsilon_2 = v_{2,2} + \frac{1}{2}w_{,2}^2,$$
$$\omega = v_{1,2} + v_{2,1} + w_{,1}w_{,2},$$
$$k_1 = -w_{,11}, \quad k_2 = -w_{,22}, \quad \tau = -w_{,12}. \tag{57}$$

which, using Eq. (51), are readily shown to coincide with von Kármán's formulae, see, e.g., [1, 18].

It should be noted that important application of the non-linear homogenized plate model developed in this paper is in the analysis of the elastic stability of composite reinforced plates. Although the mathematical analysis of this kind usually involves stability equations (expressed in terms of forces and moments) and the usual mid-surface strain expressions, one must employ the elastic relations (50) and (52) to describe the relationships between the forces, moments and strains. The problem can be considerably simplified, however, by noting that the products $w_{,\theta}\varepsilon_{\lambda\mu}$ and $w_{,\theta}\tau_{\lambda\mu}$ are smaller than other terms and may therefore be dropped from the Eqs. (50), (52). Note also that the above approach to the stability problem is based on the assumption that, at the moment of buckling, the stress and strain characteristics of the body vary on a length scale longer than the tangential dimensions of the unit cell (which are, of course, the same order of magnitude as the thickness of the layer).

7 Applications of homogenization model to the analysis of laminated plate

We will illustrate the developed non-linear asymptotic homogenization model by an example of a laminated plate of constant thickness. We will assume that all layers are made of homogeneous materials and are perfectly bonded with one another. As shown in the Fig. 2 each layer of the unit cell is completely determined by the parameters $\delta_1, \delta_2, \ldots \delta_M$ where M is the total number of layers. The thickness of the m^{th}-layer is therefore $\delta_m - \delta_{m-1}$ with $\delta_0 = 0$ and $\delta_M = 1$. The real thickness of the m^{th}-layer as measured in the original (x_1, x_2, x_3) coordinate system is $\delta(\delta_m - \delta_{m-1})$ where δ is the thickness of the laminate (again with respect to the original coordinate system).

Clearly, since material coefficients for this problem are independent of y_1 and y_2, then all partial derivatives in Eqs. (34)–(36) become ordinary derivatives with respect to z. The operators given by Eq. (12) now become

$$L_{ijn} = C_{ijn3}\frac{d}{dz}. \tag{58}$$

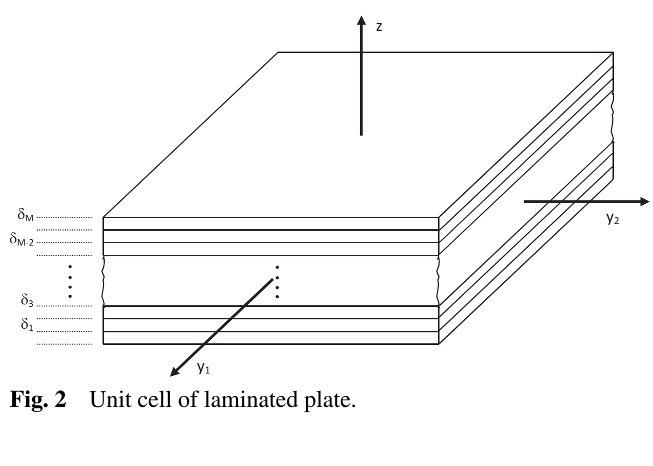

Fig. 2 Unit cell of laminated plate.

As well, we observe from Fig. 2 that

$$z^{\pm} = \pm 0.5, \quad \mathbf{n} = (0, 0, 1). \tag{59}$$

The local problems given by Eqs. (34)–(36) can be solved analytically. And substitution of the calculated local functions $b_{ij}^{\lambda\mu}$, $b_{ij}^{*\lambda\mu}$, $q_{ij}^{\alpha\lambda\mu}$, $r_{ij}^{\alpha\lambda\mu}$ into the Eq. (39) yields distribution of stresses in the laminated plate.

Omitting bulky calculations, we will illustrate this solution by calculating the stresses in an 8-layer $[+45/-45]_4$ anti-symmetric angle-ply laminate consisting of 0.125 mm thick AS/3501 graphite/epoxy laminae with material properties shown in Table 1, and subjected to forces $N_x = 10$ kN/m, $N_y = -5$ kN/m and moments $M_x = 4$ Nm/m and $M_y = -3$ Nm/m.

Table 1 Elastic material properties of laminae.

Material	E_1(GPa)	E_2(GPa)	G_{12}(GPa)	ν_{12}
AS/3501 graphite/epoxy	138	9.0	6.9	0.3

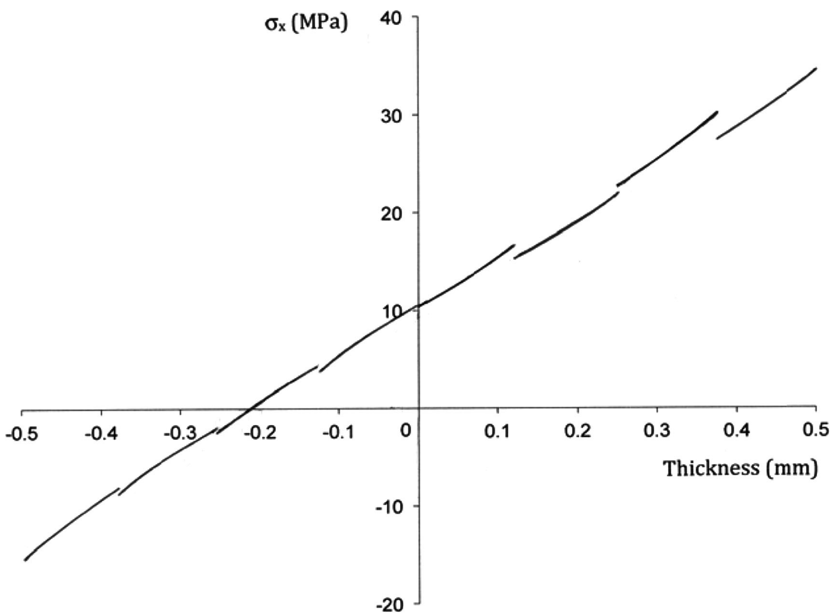

Fig. 3 Variation of stress σ_x through thickness of laminate.

The plot of stress σ_x through thickness of laminate is shown in Fig. 3. It clearly demonstrates non-linear distribution of stress resulted from accounting for the non-linearity of the problem under study.

Conclusions

The modified asymptotic homogenization method is applied to develop the geometrically non-linear homogenization model for a thin 3D composite layer with wavy surfaces. A set of 3D local unit cell problems is derived. Unlike classical homogenization schemes, the derived unit cell problems are shown to depend on boundary conditions in the transverse direction rather than periodicity. The solution of the local unit cell problems yields a set of functions which, when averaged over the volume of the unit cell, can be used to determine the effective stiffness moduli of the non-linear homogenized plate. The effective stiffnesses are substituted into the derived governing equations of the homogenized model, which in turn yield a set of local functions. These functions allow making very accurate predictions concerning 3D local mechanical stress and displacement fields.

It is seen that first type of the derived local problems coincides with local problems obtained earlier in the framework of linear elasticity theory. The second type of the derived local problems is principally new. These local problems are responsible for the non-linearity of the problem analyzed in the present paper.

The local problems are expressed in a form that shows that they are completely determined by the geometrical and material characteristics of the unit cell of the layer and are independent of the global formulation of the original boundary-value problem. It follows that derived effective stiffness moduli are universal in nature and may be used to analyze different types of boundary-value problems associated with a given composite structure.

The asymptotic homogenization model developed in the present paper is applicable to the analysis of thin composite layers for which the periodic variations of material and geometrical inhomogeneities in both tangential directions have the same order of smallness as the small transversal thickness of the layer. This type of thin-walled composite structures is very common in numerous practical applications.

The important application of the developed geometrically non-linear asymptotic homogenization plate model is to analyze the elastic stability of composite reinforced plates.

It is shown that in the limiting case of a layer of constant thickness made from the homogeneous isotropic material the derived homogenization model converges to the geometrically non-linear mean-flexure plate theory. And the obtained expressions for the mid-surface strains converge to von Kármán's formulae.

The derived non-linear homogenization model is illustrated by an example of a laminated plate. The local problems are solved in case of laminated plate, and the stresses are calculated for the 8-layer anti-symmetric angle-ply laminate consisting of graphite-epoxy laminae. The plot of stress σ_x through thickness of laminate clearly demonstrates non-linear distribution of stress resulted from accounting for the non-linearity of the problem under study.

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