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Asymptotic Analysis of Fiber-Reinforced Composites of Hexagonal Structure

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The fiber-reinforced composite materials with periodic cylindrical inclusions of a circular cross-section arranged in a hexagonal array are analyzed. The governing analytical relations of the thermal conductivity problem for such composites are obtained using the asymptotic homogenization method. The lubrication theory is applied for the asymptotic solution of the unit cell problems in the cases of inclusions of large and close to limit diameters, and for inclusions with high conductivity. The lubrication method is further generalized to the cases of finite values of the physical properties of inclusions, as well as for the cases of medium-sized inclusions. The analytical formulas for the effective coefficient of thermal conductivity of the fiber-reinforced composite materials of a hexagonal structure are derived in the cases of small conductivity of inclusions, as well as in the cases of extremely low conductivity of inclusions. The three-phase composite model (TPhM) is applied for solving the unit cell problems in the cases of the inclusions with small diameters, and the asymptotic analysis of the obtained solutions is performed for inclusions of small sizes. The obtained results are analyzed and illustrated graphically, and the limits of their applicability are evaluated. They are compared with the known numerical and asymptotic data in some particular cases, and very good agreement is demonstrated.

Keywords: Multiscale asymptotic homogenization; fiber-reinforced composite; effective coefficient of thermal conductivity; lubrication theory; three-phase composite model.

1. Introduction

Analysis of the properties of composite materials is a comprehensive and versatile problem, the solution of which can be approached with a certain degree of generality from different physical positions and various mathematical interpretations, see Ref. 1. A variety of physical characteristics and structural properties of composites leads to the need to develop the appropriate micromechanical models for the different types of composite materials which take into account characteristics of processes occurring in them on the micro- and macro-levels.

The multiscale homogenization modeling approaches have been developed for the analysis of heterogeneous materials, see e.g. Refs. 2–5.

Very effective mathematical tool for solving problems of inhomogeneous composite materials of a regular structure is the two-scale asymptotic homogenization method, see Refs. 6–12. However, the effectiveness of the asymptotic homogenization method depends essentially on the correct solution of the local unit cell problems. The practical realization of the asymptotic homogenization is possible only in combination with development of the adequate techniques of solution of local unit cell problems based on application of asymptotic methods and simplifying physical assumptions, see e.g. Refs. 9–12.

Analysis of fiber-reinforced composite materials of a hexagonal structure is presented in a number of publications from different perspectives and in different formulations of the problem. For example, Perrins *et al.*[13] investigated the effective conductivity of the 2D composite structure with a regular array of highly conductive circular cylindrical inclusions. Solution of the problem is built on the basis of analytical relations derived using the Rayleigh method, combined with the numerical analysis.

Defining relations for a hexagonal composite structure on the basis of the asymptotic homogenization obtained in Ref. 12.

The formula for the effective conductivity of a composite material with cylindrical inclusions is obtained in Ref. 14. This formula coincides with the Hashin–Shtrikman boundary, see Refs. 15 and 16, which works well in a large range of geometric sizes of the inclusions and their conductivity, except cases of contacting inclusions and their high conductivity.

The effective properties of tightly packed stochastic composites are studied in Refs. 17–19 with the use of variational methods and assuming a small parameter measuring a small distance between the inclusions. In particular, Berlyand and Novikov[18] applied this approach to derive an asymptotic formula for the effective conductivity of the composite material of hexagonal structure with circular superconducting inclusions when distance between them tends to zero.

Application of variational methods for the determination of effective moduli of polycrystalline hexagonal structure is shown in Ref. 20.

The boundaries that refine the Hashin–Shtrikman bounds for polycrystals with hexagonal symmetry are found in Ref. 21.

Eischen and Torquato[22] calculated the bounds for the volumetric and shear moduli for hexagonal arrays of circular inclusions and shown that these bounds are in a good agreement with the numerical results.

A number of publications are on the analysis of the effective properties of various composites using mathematical technique of Padé approximants, see Refs. 23–26 Padé-type approximants, see Refs. 27 and 28, and elliptic functions, see Ref. 29.

The various asymptotic techniques got a widespread applications in the analysis of composites, see Refs. 30, 31, 12, 32, 33, 14 and 34, as well as approaches involving the use of simplifying models and schemes: lubrication theory, two-phase and three-phase composite models and their modifications, etc., see Refs. 35–40, 23, 24.

In this paper, the fiber-reinforced composite materials with periodic cylindrical inclusions of a circular cross-section arranged in a hexagonal array are analyzed. The governing analytical relations of the thermal conductivity problem for such composites are derived using the asymptotic homogenization method. The following results are obtained in this paper:

- the asymptotic solutions of the local unit cell problems for the inclusions of large and close to limit diameters, and for inclusions with high conductivity are obtained using the theory of lubrication;
- the lubrication theory method is generalized to the cases of finite values of the physical properties of inclusions, as well as for the cases of medium-sized inclusions;
- the analytical relations for the effective properties of the fiber-reinforced composite materials of a hexagonal structure are derived in the cases of small physical characteristics of inclusions, as well as in the cases of extremely low conductivity of inclusions;
- the three-phase composite model (TPhM) is applied for solving the local unit cell problems in the cases of the inclusions with small diameters, and the asymptotic analysis of the obtained solutions is performed for inclusions of small sizes;
- the graphical illustration of the obtained results is presented, and the comparison with the known numerical and asymptotic data in some particular cases is performed.

2. The Problem Formulation and Derivation of a General Solution Using the Asymptotic Homogenization Method

The thermal conductivity problem for the fiber-reinforced composite material with periodically arranged hexagonal array of circular cylindrical fibers is considered, see Fig. 1.

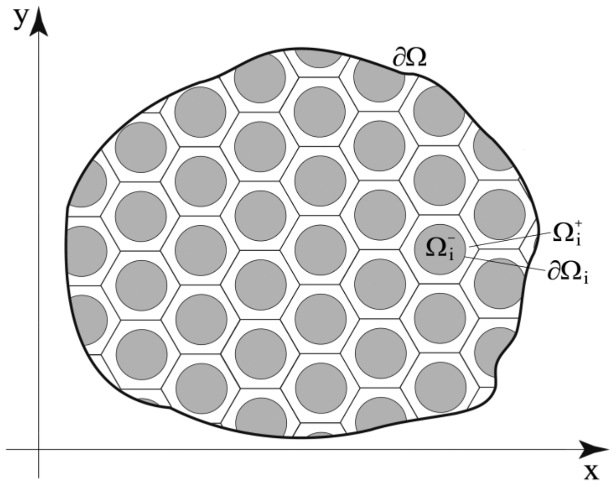

Fig. 1. Cross-section of fiber-reinforced composite material with periodically arranged hexagonal array of circular cylindrical fibers.

The thermal conductivity problem for such composite material can be written as follows:

$$\lambda^{+}\left(\frac{\partial^2 u^+}{\partial x^2}+\frac{\partial^2 u^+}{\partial y^2}\right)=F \quad \text{in } \Omega_i^+; \tag{2.1}$$

$$\lambda^{-}\left(\frac{\partial^2 u^-}{\partial x^2}+\frac{\partial^2 u^-}{\partial y^2}\right)=F \quad \text{in } \Omega_i^-; \tag{2.2}$$

$$u^+=u^-, \quad \lambda^+\frac{\partial u^+}{\partial \mathbf{n}}=\lambda^-\frac{\partial u^-}{\partial \mathbf{n}} \quad \text{on } \partial\Omega_i, \tag{2.3}$$

where u^+, u^- are the temperature distributions in the matrix and inclusions; λ_i^+, λ_i^- are thermal conductivities of matrix and fiber materials; $\lambda = \frac{\lambda^-}{\lambda^+}$; F is a density of heat sources; and $\mathbf{n}$ is an external normal vector to the boundary of inclusion.

According to the asymptotic homogenization method, see e.g. Refs. 8–10, and using the technique of two-scale expansions the solution of the problem (2.1)–(2.3) is represented in the form of asymptotic series in powers of a small dimensionless parameter $\varepsilon(\varepsilon \ll 1)$ characterizing the small dimension of the unit cell of the composite material:

$$u^{\pm}=u_0(x,\ y)+\varepsilon u_1^{\pm}(x,\ y,\ \xi,\ \eta)+\varepsilon^2 u_2^{\pm}(x,\ y,\ \xi,\ \eta)+\cdots, \tag{2.4}$$

where ξ, η are so-called "fast" variables defined as follows: $\xi = x/\varepsilon, \eta = y/\varepsilon$.

In view of Eq. (2.4) and after splitting relations (2.1)–(2.3) into the powers of small parameter ε the solution of the problem can be reduced into two stages.

In the first stage the solution of the local unit cell problem is determined. The local problem is given by Eqs. (2.5)–(2.7), and it is defined in the fast variables only

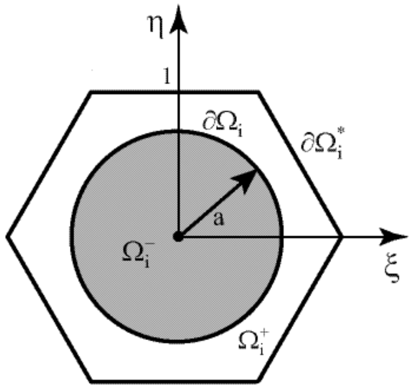

Fig. 2. Unit cell of the composite material shown in Fig. 1, Ω_i^+ is domain of matrix, and Ω_i^- is domain of the inclusion.

in the domain of the unit cell of the composite material shown in Fig. 2:

$$\frac{\partial^2 u_1^\pm}{\partial \xi^2} + \frac{\partial^2 u_1^\pm}{\partial \eta^2} = 0 \quad \text{in } \Omega_i^\pm; \tag{2.5}$$

$$u_1^+ = u_1^-, \quad \frac{\partial u_1^+}{\partial \bar{\mathbf{n}}} - \lambda \frac{\partial u_1^-}{\partial \bar{\mathbf{n}}} = (\lambda - 1)\frac{\partial u_0}{\partial \mathbf{n}} \quad \text{on } \partial\Omega_i; \tag{2.6}$$

$$u_1^+ = 0 \quad \text{on } \partial\Omega_i^*. \tag{2.7}$$

Here $\bar{\mathbf{n}}$ is an external normal vector to the boundary of inclusion in the fast variables.

Further, in the second stage, the main part of the solution $u_0(x, y)$ is determined from the homogenized equation, see Ref. 23

$$\bar{q}\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right) + \frac{1}{|\Omega_i^*|}\left[\iint_{\Omega_i^+}\left(\frac{\partial^2 u_1^+}{\partial x \partial \xi} + \frac{\partial^2 u_1^+}{\partial y \partial \eta}\right) d\xi \partial\eta \right.$$
$$\left. + \lambda \iint_{\Omega_i^-}\left(\frac{\partial^2 u_1^-}{\partial x \partial \xi} + \frac{\partial^2 u_1^-}{\partial y \partial \eta}\right) d\xi \partial\eta\right] = F, \tag{2.8}$$

where $\Omega_i^* = \Omega_i^+ \cup \Omega_i^-$; and $\bar{q} = \frac{|\Omega_i^+| + \lambda|\Omega_i^-|}{|\Omega_i^*|}$ denotes the area of the corresponding domain.

The homogenized equation in the general form can be transformed as follows taking into account the expressions u_1^+, u_1^-, defined from the solution of the unit cell problem (2.5)–(2.7):

$$q_x \frac{\partial^2 u_0}{\partial x^2} + q_y \frac{\partial^2 u_0}{\partial y^2} = F \quad \text{in } \Omega^*, \tag{2.9}$$

where q_x, q_y are the effective properties defined as follows:

$$q_x = \bar{q} + \frac{1}{|\Omega_i^*|}\left(\iint_{\Omega_i^+} \frac{\partial u_{1(1)}^+}{\partial \xi} d\xi \partial\eta + \lambda \iint_{\Omega_i^-} \frac{\partial u_{1(1)}^-}{\partial \xi} d\xi \partial\eta\right), \tag{2.10}$$

$$q_y = \bar{q} + \frac{1}{|\Omega_i^*|}\left(\iint_{\Omega_i^+} \frac{\partial u_{1(2)}^+}{\partial \eta} d\xi \partial\eta + \lambda \iint_{\Omega_i^-} \frac{\partial u_{1(2)}^-}{\partial \eta} d\xi \partial\eta\right). \tag{2.11}$$

Here $u_{1(1)}^{\pm}$ and $u_{1(2)}^{\pm}$ are the solutions of the unit cell problem (2.5)–(2.7) in fast variables defining the ε-order term $u_1^{\pm}(x,\ y,\ \xi,\ \eta)$ in the asymptotic expansion for the temperature distribution (2.4):

$$u_1^{\pm} = u_{1(1)}^{\pm}(\xi,\ \eta)\frac{\partial u_0}{\partial x} + u_{1(2)}^{\pm}(\xi,\ \eta)\frac{\partial u_0}{\partial y}. \tag{2.12}$$

Thus, in solving the thermal conductivity problem for the fiber-reinforced composite material with periodically arranged hexagonal array of circular cylindrical fibers, the key task is in solving the local unit cell problem (2.5)–(2.7) in order to calculate the effective properties of the composite material given by Eqs. (2.10) and (2.11).

In the sequel the appropriate analytical methods of solving the local unit cell problem will be developed, taking into account the geometrical and physical characteristics of the composite material and its hexagonal structure.

3. Application of the Lubrication Theory for Solving the Unit Cell Problem

The proposed approach is based on the combined use of asymptotic homogenization and the lubrication theory, see Refs. 36, 23 and 39. Due to its physical nature, this approach is applicable to the asymptotic analysis of composite materials with inclusions of large sizes, with fibers of radii close to the limit ($a \to 1$), and with very high conductivity ($\lambda \to \infty$).

The general idea of the lubrication theory is to replace the boundary-value problem in the original domain with a problem formulated in the domain with more simple geometry. Next, the transformed geometrical parameter of the unit cell is considered to be a variable function of the coordinates.

This approach is applied in this section to determine the effective coefficient of thermal conductivity q_y in the direction of y-axis (or η-axis in fast variables). For this purpose, the external hexagonal contour of the unit cell is replaced by a circle of radius b, see Fig. 3.

For the circular geometry shown in Fig. 3 it is convenient to use the fast polar coordinates r, θ, and represent the unit cell problem (2.5)–(2.7) in these polar

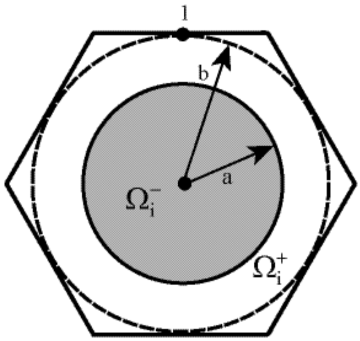

Fig. 3. The computational model of the lubrication theory for the fiber-reinforced composite of hexagonal structure.

coordinates as follows:

$$\frac{\partial^2 u_1^+}{\partial r^2} + \frac{1}{r}\frac{\partial u_1^+}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_1^+}{\partial \theta^2} = 0 \quad \text{in } \Omega_i^+; \tag{3.1}$$

$$\frac{\partial^2 u_1^-}{\partial r^2} + \frac{1}{r}\frac{\partial u_1^-}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_1^-}{\partial \theta^2} = 0 \quad \text{in } \Omega_i^-; \tag{3.2}$$

$$u_1^+ = u_1^-; \quad \frac{\partial u_1^+}{\partial r} - \lambda\frac{\partial u_1^-}{\partial r} = (\lambda - 1)\left(\frac{\partial u_0}{\partial x}\cos\theta + \frac{\partial u_0}{\partial y}\sin\theta\right) \quad \text{for } r = a; \tag{3.3}$$

$$u_1^+ = 0 \quad \text{for } r = b. \tag{3.4}$$

The problem (3.1)–(3.4) has the following solution:

$$u_1^- = A_1 r\cos\theta + A_2 r\sin\theta; \tag{3.5}$$

$$u_1^+ = \left(B_1 r + \frac{C_1}{r}\right)\cos\theta + \left(B_2 r + \frac{C_2}{r}\right)\sin\theta, \tag{3.6}$$

where $A_i,\ B_i,\ C_i\ (i = 1,\ 2)$ are constants that can be found from the interface conditions (3.3) and the boundary condition (3.4) as follows:

$$\begin{aligned} A_1 &= -\frac{(\lambda-1)(b^2-a^2)}{(b^2+a^2)+\lambda(b^2-a^2)}\frac{\partial u_0}{\partial x}; \\ B_1 &= \frac{(\lambda-1)a^2}{(b^2+a^2)+\lambda(b^2-a^2)}\frac{\partial u_0}{\partial x}; \\ C_1 &= -\frac{(\lambda-1)a^2b^2}{(b^2+a^2)+\lambda(b^2-a^2)}\frac{\partial u_0}{\partial x}; \\ A_2 &\to A_1;\ B_2 \to B_1;\ C_2 \to C_1\ \left(\frac{\partial u_0}{\partial x} \to \frac{\partial u_0}{\partial y}\right). \end{aligned} \tag{3.7}$$

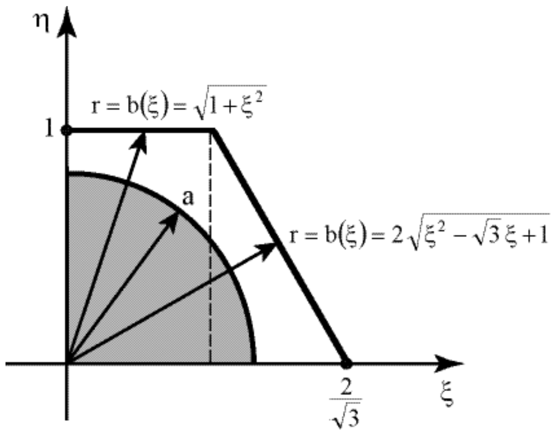

Fig. 4. Approximation of the hexagonal contour of the unit cell.

Continuing solution in the framework of the lubrication theory, the external contour of unit cell is further considered as a circle of variable radius $b(\xi)$:

$$b = \begin{cases} b(\xi) = \sqrt{1+\xi^2} & \text{for } 0 \le \xi \le \dfrac{1}{\sqrt{3}}, \\ b(\xi) = 2\sqrt{\xi^2 - \sqrt{3}\xi + 1} & \text{for } \dfrac{1}{\sqrt{3}} \le \xi \le \dfrac{2}{\sqrt{3}}. \end{cases} \tag{3.8}$$

Figure 4 shows one-quarter of the unit cell over which the integration is performed in the further realization of the lubrication approach.

In determining the effective thermal conductivity q_y in the direction of y-axis the integration in Eq. (2.8) is performed over the original area of the hexagonal unit cell on account of Eq. (3.8), i.e., by considering $A_2,\ B_2,\ C_2$ as functions in the variable ξ.

The following asymptotic expression for the effective thermal conductivity $q_y^{(\infty)}$ is found as a function $q_y^{(\infty)} = q(a)$ of the radius of fiber a for large sizes of inclusions:

$$\begin{aligned} q_y^{(\infty)} &= \frac{2\sqrt{3}a^2}{\sqrt{1-a^2}} \arctan \frac{\sqrt{3}}{3\sqrt{1-a^2}} + 1 + \frac{\sqrt{3}a^2}{3}\left(\frac{\pi}{4} - \frac{3}{2}\arcsin\frac{\sqrt{3}}{3a}\right) \\ &\quad + \frac{4\sqrt{3}a^2}{3\sqrt{1-a^2}}\left[\arctan\left((\sqrt{3}a - \sqrt{3a^2-1})\sqrt{\frac{1-a}{1+a}}\right)\right. \\ &\quad - \frac{1}{4}\arctan\frac{2(\sqrt{3}a - 1 - \sqrt{3a^2-1})\sqrt{1-a^2}}{(1+\sqrt{3})(1-\sqrt{3}a)a + \sqrt{3a^2-1}(a-2+\sqrt{3}a)+2} \\ &\quad \left. - \frac{1}{8}\arctan\frac{2\sqrt{1-a^2}}{a}\right] - \frac{a^2}{4}\ln\frac{(2+3a^2+2\sqrt{3(3a^2-1)})}{(4-3a^2)}. \end{aligned} \tag{3.9}$$

In the limiting case $a \to 1$, Eq. (3.9) can be transformed to the following asymptotic expression on account of the main and first-order terms of the asymptotic expansion:

$$q_{y\ \mathrm{Asympt}}^{(\infty)} = \frac{\sqrt{3}\pi}{\sqrt{1-a^2}} + 1 + \frac{\sqrt{3}\pi}{12} - \frac{\sqrt{3}}{2}\arcsin\frac{\sqrt{3}}{3} - \frac{1}{4}\ln(5+2\sqrt{6}) - \sqrt{3}(\sqrt{3}+\sqrt{2}). \tag{3.10}$$

Note that the main term of the asymptotic expression (3.10)

$$q_{\mathrm{Asympt}} = \frac{\sqrt{3}\pi}{\sqrt{1-a^2}} \tag{3.11}$$

coincides with the asymptotic formula obtained in Ref. 18 by means of generalization of Keller[41] method for a square array of fibers to the case of hexagonal array of fibers.

4. The Interrelation of the Effective Coefficients of Thermal Conductivity in Different Directions of Coordinate Axes

Asymptotic behavior of the effective coefficient of thermal conductivity q_x in the x-axis direction in the case of large inclusions, with fiber radius close to a limit ($a \to 1$), and with very large conductivity of fiber material ($\lambda \to \infty$), can be analyzed by considering the heat flux $I(\lambda, a)$ in the unit cell. The flow distribution in one-quarter of the unit cell is shown in Fig. 5.

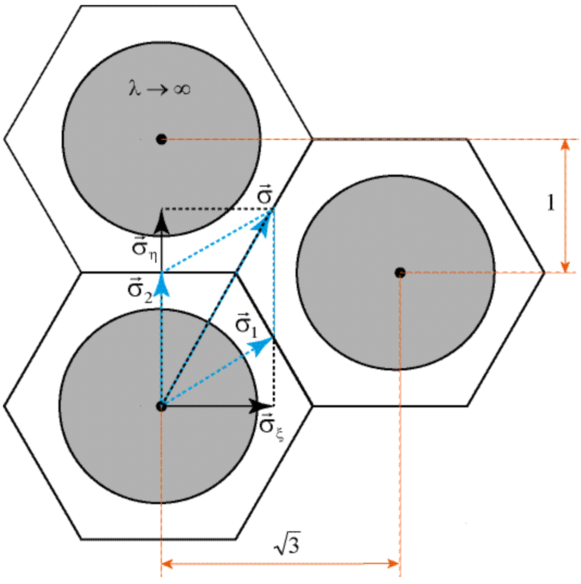

Fig. 5. The heat flux distribution in one-quarter of the unit cell for $\lambda \to \infty, a \gg 0$.

Since heat flux intensity $I(\lambda,\ a)$ is similar in the directions $\boldsymbol{\sigma}_1(\frac{\sqrt{3}}{2};\ \frac{1}{2})$ and $\boldsymbol{\sigma}_2(0;\ 1)$, the resulting heat flux $\boldsymbol{\sigma}$ can be determined as follows:

$$I(\lambda,\ a)\boldsymbol{\sigma}_1\left(\frac{\sqrt{3}}{2};\ \frac{1}{2}\right) + I(\lambda,\ a)\boldsymbol{\sigma}_2(0;\ 1) = \boldsymbol{\sigma}\left(\frac{\sqrt{3}}{2}I(\lambda,\ a);\ \frac{3}{2}I(\lambda,\ a)\right). \qquad (4.1)$$

Then the projections of $\boldsymbol{\sigma}$ onto the ξ- and η-axes are the following:

$$\boldsymbol{\sigma}_\xi\left(\frac{\sqrt{3}}{2}I(\lambda,\ a);\ 0\right) \quad \text{and} \quad \boldsymbol{\sigma}_\eta\left(0;\ \frac{3}{2}I(\lambda,\ a)\right). \qquad (4.2)$$

Therefore, due to the geometric structure of hexagonal lattice (see Fig. 5) the effective coefficients of conductivity will be similar in the directions of x- and y-axes:

$$q_x^{(\infty)} = q_y^{(\infty)} = q^{(\infty)}. \qquad (4.3)$$

Thus the homogenized medium will be isotropic.

Keller[41,42] has proved a theorem that relates the average conductivities of the arrays of cylinders in x-axis and y-axis directions. The conditions of this theorem are fulfilled in the considered case, and we have

$$q_y(\lambda^{-1}) = q_x^{-1}(\lambda). \qquad (4.4)$$

The generalization of this relation is given in Ref. 13, where Eq. (4.4) is also proved for the composite material with hexagonal array of circular cylindrical fibers.

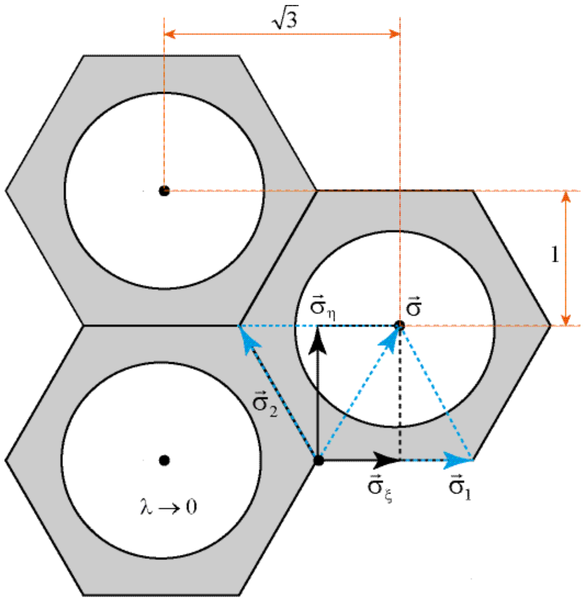

Fig. 6. The heat flux distribution in one-quarter of the unit cell for $\lambda \to 0, a \gg 0$.

Equations (4.3) and (4.4) yield the following interrelation of asymptotic representations of the effective coefficients of thermal conductivity for the fiber-reinforced composite with hexagonal structure in the directions of x- and y-axes in the case of very small conductivities of fibers ($\lambda \to 0$):

$$q_x^{(0)} = q_y^{(0)} = q^{(0)}. \tag{4.5}$$

The geometrical interpretation of the flows defining the effective coefficients satisfying Eq. (4.5) is given in Fig. 6.

It follows from Fig. 6 that $\boldsymbol{\sigma}_1(\frac{2\sqrt{3}}{3};\ 0)$ and $\boldsymbol{\sigma}_2(-\frac{\sqrt{3}}{3};\ 1)$, and therefore,

$$I(\lambda,\ a)\boldsymbol{\sigma}_1\left(\frac{2\sqrt{3}}{3};\ 0\right) + I(\lambda,\ a)\boldsymbol{\sigma}_2\left(-\frac{\sqrt{3}}{3};\ 1\right) = \boldsymbol{\sigma}\left(\frac{\sqrt{3}}{3}I(\lambda,\ a);\ I(\lambda,\ a)\right) \tag{4.6}$$

and

$$\boldsymbol{\sigma}\left(\frac{\sqrt{3}}{3}I(\lambda,\ a);\ I(\lambda,\ a)\right) = \boldsymbol{\sigma}_\xi\left(\frac{\sqrt{3}}{3}I(\lambda,\ a);\ 0\right) + \boldsymbol{\sigma}_\eta(0;\ I(\lambda,\ a)), \tag{4.7}$$

which justifies Eq. (4.5).

5. Generalization of the Lubrication Theory to the Case of Finite Values of Thermal Conductivity of Fibers

Equation (3.9) for the effective coefficient of thermal conductivity is derived for the case of $\lambda \to \infty$. It can be shown that the lubrication theory approach can be generalized to the case of finite values $\lambda \gg 1$. In this case application of the lubrication approximation similarly to the above scheme leads to the following expressions for the effective coefficient:

(i) In the case $\lambda \gg 1, a \gg 1$:

$$\begin{aligned} q = {} & \frac{2\sqrt{3}\gamma a^2}{\sqrt{1-\gamma a^2}} \arctan \frac{\sqrt{3}}{3\sqrt{1-\gamma a^2}} + 1 + \frac{\gamma a^2\sqrt{3}}{3}\left\{\frac{\pi}{4} - \frac{3}{2}\arcsin\frac{\sqrt{3}}{3\sqrt{\gamma}a}\right. \\ & + \frac{4}{\sqrt{1-\gamma a^2}}\left[\arctan\left((\sqrt{3\gamma}a - \sqrt{3\gamma a^2 - 1})\frac{\sqrt{1-\sqrt{\gamma}a}}{\sqrt{1+\sqrt{\gamma}a}}\right)\right. \\ & - \frac{1}{4}\arctan\frac{2(\sqrt{3\gamma}a - 1 - \sqrt{3\gamma a^2 - 1})\sqrt{1-\gamma a^2}}{(1+\sqrt{3})(1-\sqrt{3\gamma}a)\sqrt{\gamma}a + \sqrt{3\gamma a^2 - 1}(\sqrt{\gamma}a - 2 + \sqrt{3\gamma}a) + 2} \\ & \left.\left. - \frac{1}{8}\arctan\left(2\sqrt{\frac{1}{\gamma a^2} - 1}\right)\right] - \frac{\sqrt{3}}{4}\ln\frac{(2 + 3\gamma a^2 + 2\sqrt{3(3\gamma a^2 - 1)})}{(4 - 3\gamma a^2)}\right\}, \end{aligned} \tag{5.1}$$

where

$$\gamma = \frac{\lambda - 1}{\lambda + 1} \tag{5.2}$$

and domain of applicability is defined as $\sqrt{\gamma}a \geq \frac{1}{\sqrt{3}}$;

(ii) In the case $\lambda \gg 1, a \to 1$:

$$q = \frac{\sqrt{6\lambda}\pi}{2} + 1 + \frac{\sqrt{3}\pi}{12} - \frac{\sqrt{3}}{2}\arcsin\frac{\sqrt{3}}{3} - \frac{1}{4}\ln(5+2\sqrt{6}) - \sqrt{3}(\sqrt{3}+\sqrt{2}), \tag{5.3}$$

and domain of applicability is defined as $\lambda \geq 2$.

Note that Eq. (5.1) coincides with the previously obtained Eq. (3.9) when taking the limit $\lambda \to \infty$ in Eqs. (5.1) and (5.2).

6. Generalization of the Lubrication Theory to the Case of Medium-Sized Inclusions

Note that the above asymptotic expression (3.9) for the effective coefficient of thermal conductivity was obtained for the large inclusions that were close to the limit ($a = 1$), and the domain of its applicability was limited by $a \geq \frac{1}{\sqrt{3}} \approx 0.577$. In this connection it is interesting to generalize the proposed lubrication theory approach to the case of medium-sized inclusions $a \leq \frac{1}{\sqrt{3}}$, see Fig. 7.

Using Eqs. (3.5)–(3.7) and taking into account Eq. (3.8), one obtains in this case the following expression for the effective coefficient of thermal conductivity $q_{\text{Med.Incl.}}$ of the composite with very large conductivity $\lambda \to \infty$ of a medium-sized inclusions $a \leq \frac{1}{\sqrt{3}}$:

$$q_{\text{Med.Incl.}} = 1 + \frac{\sqrt{3}a^2}{\sqrt{1-a^2}}\arctan\frac{\sqrt{3}}{3\sqrt{1-a^2}} + \frac{\sqrt{3}a^2}{3}\left[\frac{2}{\sqrt{1-a^2}}\left(2\arctan\sqrt{\frac{1-a}{1+a}} + \arctan\frac{a}{\sqrt{1-a^2}}\right) - \frac{\pi}{2}\right]. \tag{6.1}$$

This approach can be further developed for the case of large but finite conductivity of fibers $\lambda \gg 1$ and medium-sized inclusions. In this case, the following expression

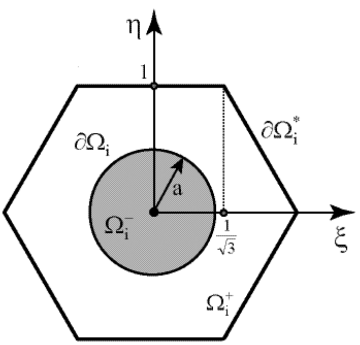

Fig. 7. Unit cell of fiber-reinforced composite of hexagonal structure with the medium-sized inclusions.

is obtained:

$$q_{\text{Med.Incl.}} = 1 + \frac{\sqrt{3}\gamma a^2}{\sqrt{1-\gamma a^2}} \arctan \frac{\sqrt{3}}{3\sqrt{1-\gamma a^2}} + \frac{\sqrt{3}\gamma a^2}{3}\left[\frac{2}{\sqrt{1-\gamma a^2}}\left(2\arctan\frac{\sqrt{1-\sqrt{\gamma}a}}{\sqrt{1+\sqrt{\gamma}a}} + \arctan\frac{\sqrt{\gamma}a}{\sqrt{1-\gamma a^2}}\right) - \frac{\pi}{2}\right], \tag{6.2}$$

and the domain of applicability is defined as $\sqrt{\gamma}a = \sqrt{\frac{\lambda-1}{\lambda+1}}a \le \frac{1}{\sqrt{3}}$.

7. Asymptotic Expressions for the Effective Coefficients of Thermal Conductivity in the Case of Very Small Conductivity of Inclusions

It is possible to derive the asymptotic expressions for the effective coefficients of thermal conductivity in the case of very small conductivity of inclusions ($\lambda \to 0$), using the above obtained solutions (3.9), (3.10), (6.1) and the relations (4.2), (4.3).

(i) In case $\lambda \to 0, a \ge \frac{1}{\sqrt{3}}$:

$$q^{(0)} = \left\{\frac{2\sqrt{3}a^2}{\sqrt{1-a^2}} \arctan\frac{\sqrt{3}}{3\sqrt{1-a^2}} + 1 + \frac{\sqrt{3}a^2}{3}\left(\frac{\pi}{4} - \frac{3}{2}\arcsin\frac{\sqrt{3}}{3a}\right) + \frac{4\sqrt{3}a^2}{3\sqrt{1-a^2}}\left[\arctan\left((\sqrt{3}a - \sqrt{3a^2-1})\sqrt{\frac{1-a}{1+a}}\right) - \frac{1}{4}\arctan\frac{2(\sqrt{3}a - 1 - \sqrt{3a^2-1})\sqrt{1-a^2}}{(1+\sqrt{3})(1-\sqrt{3}a)a + \sqrt{3a^2-1}(a-2+\sqrt{3}a)+2} - \frac{1}{8}\arctan\frac{2\sqrt{1-a^2}}{a}\right] - \frac{a^2}{4}\ln\frac{(2+3a^2+2\sqrt{3(3a^2-1)})}{(4-3a^2)}\right\}^{-1}. \tag{7.1}$$

(ii) In case $\lambda \to 0, a \to 1$:

$$q^{(0)}_{\text{Asympt}} = \frac{\sqrt{1-a^2}}{\sqrt{3}\pi + \sqrt{1-a^2}\left(1 + \frac{\sqrt{3}\pi}{12} - \frac{\sqrt{3}}{2}\arcsin\frac{\sqrt{3}}{3} - \frac{1}{4}\ln(5+2\sqrt{6}) - \sqrt{3}(\sqrt{3}+\sqrt{2})\right)}. \tag{7.2}$$

(iii) In case $\lambda \to 0, a \leq \frac{1}{\sqrt{3}}$:

$$q_{\text{Med.Incl.}} = \left\{1 + \frac{\sqrt{3}a^2}{\sqrt{1-a^2}} \arctan \frac{\sqrt{3}}{3\sqrt{1-a^2}} + \frac{\sqrt{3}a^2}{3} \times \left[\frac{2}{\sqrt{1-a^2}}\left(2\arctan\sqrt{\frac{1-a}{1+a}} + \arctan\frac{a}{\sqrt{1-a^2}}\right) - \frac{\pi}{2}\right]\right\}^{-1}. \tag{7.3}$$

8. The Generalizing Analytical Expressions for the Effective Coefficients in the Case of Small Conductivity of Inclusions

Applying approach developed in Sec. 4 and using Eq. (4.4), it is possible to generalize the obtained results and to derive the analytical expressions for the effective coefficients of fiber-reinforced composite of hexagonal structure for the small conductivity of fibers.

Taking into account Eqs. (5.1)–(5.3) and (6.2) we obtain the following formulae.

(i) In the case of large inclusions $a \gg 0$ with small conductivity $\lambda \ll 1$:

$$q = \left\{-\frac{2\sqrt{3}\gamma a^2}{\sqrt{1+\gamma a^2}}\text{arctg}\frac{\sqrt{3}}{3\sqrt{1+\gamma a^2}} + 1 - \frac{\gamma a^2\sqrt{3}}{3}\left[\frac{\pi}{4} - \frac{3}{2}\arcsin\frac{\sqrt{3}}{3\sqrt{-\gamma}a} + \frac{4}{\sqrt{1+\gamma a^2}}\left(\arctan\left((\sqrt{-3\gamma}a - \sqrt{-3\gamma a^2 - 1})\frac{\sqrt{1-\sqrt{-\gamma}a}}{\sqrt{1+\sqrt{-\gamma}a}}\right) - \frac{1}{4}\arctan\frac{2(\sqrt{-3\gamma}a - 1 - \sqrt{-3\gamma a^2 - 1})\sqrt{1+\gamma a^2}}{(1+\sqrt{3})(1+\sqrt{-3\gamma}a)\sqrt{-\gamma}a + \sqrt{-3\gamma a^2 - 1}(\sqrt{-\gamma}a - 2 + \sqrt{-3\gamma}a) + 2} - \frac{1}{8}\arctan\left(2\sqrt{-\frac{1}{\gamma a^2} - 1}\right)\right) - \frac{\sqrt{3}}{4}\ln\frac{(2 - 3\gamma a^2 + 2\sqrt{-3(3\gamma a^2 + 1)})}{(4 + 3\gamma a^2)}\right]\right\}^{-1}, \tag{8.1}$$

and the domain of applicability is defined as $\sqrt{-\gamma}a = \sqrt{\frac{1-\lambda}{\lambda+1}}a \geq \frac{1}{\sqrt{3}}$;

(ii) In the case of close to limit large inclusions $a \to 1$ with small conductivity $\lambda \ll 1$:

$$q = \frac{12\sqrt{\lambda}}{6\sqrt{6}\pi + \left(12 + \sqrt{3}\pi - 6\sqrt{3}\arcsin\frac{\sqrt{3}}{3} - 3\ln(5 + 2\sqrt{6}) - 12\sqrt{3}(\sqrt{3} + \sqrt{2})\right)\sqrt{\lambda}}, \tag{8.2}$$

and the domain of applicability is defined as $\lambda \leq 0.5$;

(iii) In the case of medium-size inclusions $a \ll 1$ with small but finite conductivity $\lambda \ll 1$, given that $\sqrt{-\gamma}a = \sqrt{\frac{1-\lambda}{\lambda+1}}a \leq \frac{1}{\sqrt{3}}$:

$$q_{\text{Med.Incl.}} = \left\{1 - \frac{\sqrt{3}\gamma a^2}{\sqrt{1+\gamma a^2}}\text{arctg}\frac{\sqrt{3}}{3\sqrt{1+\gamma a^2}} - \frac{\sqrt{3}\gamma a^2}{3}\left[\frac{2}{\sqrt{1+\gamma a^2}}\right.\right.$$
$$\left.\left.\times\left(2\text{arctg}\frac{\sqrt{1-\sqrt{-\gamma}a}}{\sqrt{1+\sqrt{-\gamma}a}} + \text{arctg}\frac{\sqrt{-\gamma}a}{\sqrt{1+\gamma a^2}}\right) - \frac{\pi}{2}\right]\right\}^{-1}. \quad (8.3)$$

9. Application of the TPhM for Solving Unit Cell Problems in the Case of Small Size of Inclusions

TPhM is based on the replacement of the entire periodic composite structure, with the exception of one unit cell, by a homogeneous medium with some unknown and sought effective characteristics. Further these effective parameters are determined from the relations derived from the energy principle stating that the energies stored in the composite material and the equivalent homogeneous medium are equal, see Refs. 23, 35, 37, 39 and 14.

For fiber-reinforced composite of hexagonal structure under study in accordance with the TPhM technique, the entire periodic array of fibers with the exception of a single unit cell, is replaced by a homogeneous medium with an unknown effective coefficient $\tilde{\lambda}$ as shown in Fig. 8.

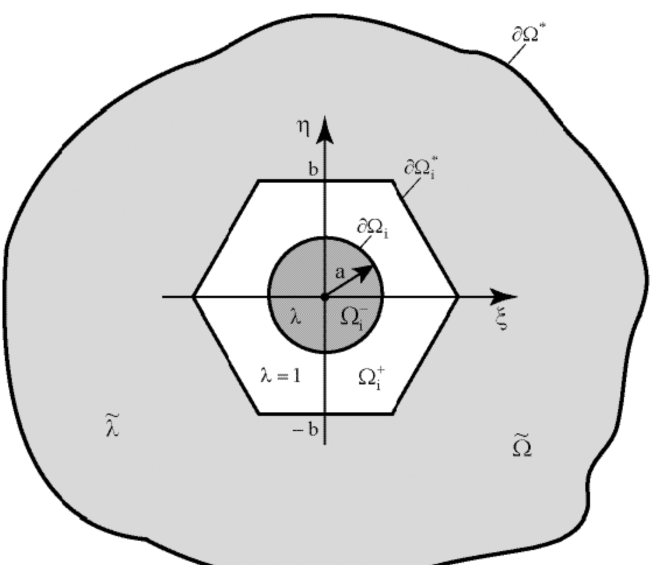

Fig. 8. Three-phase model for fiber-reinforced composite of hexagonal structure.

The TPhM solution requires the following steps:

(i) The asymptotic expansions (2.4) are complemented by expansion of the function $\tilde{u}(x, y, \xi, \eta)$ defined in the homogeneous medium $\tilde{\Omega}$ with the effective coefficient $\tilde{\lambda}$ in powers of small parameter ε:

$$\tilde{u} = u_0(x,\ y) + \varepsilon\tilde{u}_1(x,\ y,\ \xi,\ \eta) + \varepsilon^2\tilde{u}_2(x,\ y,\ \xi,\ \eta) + \cdots. \tag{9.1}$$

(ii) The boundary condition on the outer contour of the unit cell (2.7) is replaced by the perfect bonding conditions at the interface of the matrix and the equivalent homogeneous medium.

(iii) The decay conditions for the function $\tilde{u}(x, y, \xi, \eta)$ and its derivative for $\xi^2 + \eta^2 \to \infty$ are added.

As a result, the unit cell problem (2.5)–(2.7) is transformed as follows:

$$\frac{\partial^2 u_1^{\pm}}{\partial \xi^2} + \frac{\partial^2 u_1^{\pm}}{\partial \eta^2} = 0 \quad \text{in } \Omega_i^{\pm}; \tag{9.2}$$

$$\frac{\partial^2 \tilde{u}_1}{\partial \xi^2} + \frac{\partial^2 \tilde{u}_1}{\partial \eta^2} = 0 \quad \text{in } \tilde{\Omega}; \tag{9.3}$$

$$u_1^+ = u_1^-; \quad \frac{\partial u_1^+}{\partial \bar{n}} - \lambda\frac{\partial u_1^-}{\partial \bar{n}} = (\lambda - 1)\frac{\partial u_0}{\partial n} \quad \text{on } \partial\Omega_i; \tag{9.4}$$

$$u_1^+ = \tilde{u}_1; \quad \frac{\partial u_1^+}{\partial \bar{n}} - \tilde{\lambda}\frac{\partial \tilde{u}_1}{\partial \bar{n}} = (\tilde{\lambda} - 1)\frac{\partial u_0}{\partial n} \quad \text{on } \partial\tilde{\Omega}; \tag{9.5}$$

$$\begin{aligned} &\tilde{u}_1 \to 0; \quad \frac{\partial \tilde{u}_1}{\partial \xi} \to 0 \quad \text{for } \xi = \ \to \pm\infty, \\ &\tilde{u}_1 \to 0; \quad \frac{\partial \tilde{u}_1}{\partial \eta} \to 0 \quad \text{for } \eta = \ \to \pm\infty. \end{aligned} \tag{9.6}$$

In general case, in solving the unit cell problem (9.1)–(9.6) the outer contour of the unit cell $\partial\tilde{\Omega}$ in polar coordinates $r,\ \theta$ can be written as follows, see Ref. 23:

$$r = r_0 + \varepsilon_1 f(\theta), \tag{9.7}$$

where $r_0 = \text{const} > 0$; $f(\theta)$ is a function characterizing the geometry of the outer contour, and ε_1 is a small parameter: $|\varepsilon_1| = 1/15$, see Ref. 43.

Consider the zero-approximation of the solution. In this case the mathematical meaning of approximation given by Eq. (9.7) is that the hexagonal contour of unit cell is replaced by a circle. The radius of the circle r^+ is selected in a way to preserve equality of areas of the original and transformed matrix domains: $\pi(r^+)^2 = 2\sqrt{3}b^2$, and therefore,

$$r^+ = \sqrt{\frac{2\sqrt{3}}{\pi}}b. \tag{9.8}$$

Geometry of computational model of such structure is shown in Fig. 9.

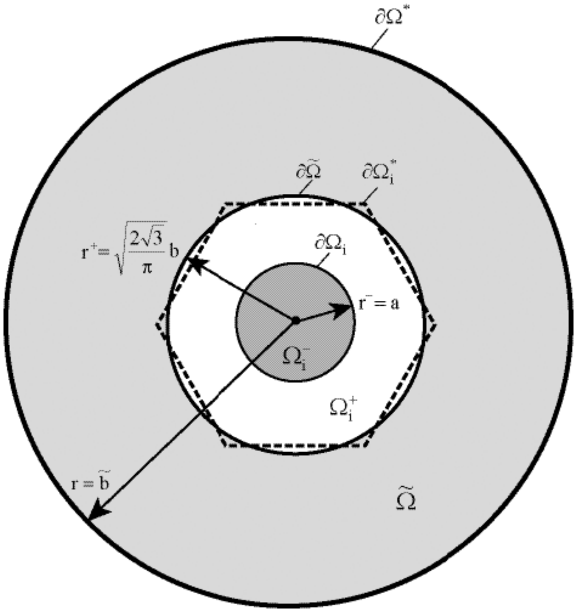

Fig. 9. Computational model for three-phase structure with hexagonal unit cell.

Unit cell problem (9.2)–(9.6) in the fast polar coordinates r, θ can be written as follows:

$$\frac{\partial^2 u_1^\pm}{\partial r^2} + \frac{1}{r}\frac{\partial u_1^\pm}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_1^\pm}{\partial \theta^2} = 0 \quad \text{in } \Omega_i^\pm; \tag{9.9}$$

$$\frac{\partial^2 \tilde{u}_1}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{u}_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{u}_1}{\partial \theta^2} = 0 \quad \text{in } \tilde{\Omega}_0; \tag{9.10}$$

$$\begin{gathered} u_1^+ = u_1^-; \\ \frac{\partial u_1^+}{\partial r} - \lambda\frac{\partial u_1^-}{\partial r} = (\lambda - 1)\left(\frac{\partial u_0}{\partial x}\cos\theta + \frac{\partial u_0}{\partial y}\sin\theta\right) \\ \text{on } \partial\Omega_i : r = r^- = a; \end{gathered} \tag{9.11}$$

$$\begin{gathered} u_1^+ = \tilde{u}_1; \\ \frac{\partial u_1^+}{\partial r} - \tilde{\lambda}\frac{\partial \tilde{u}_1}{\partial r} = (\tilde{\lambda} - 1)\left(\frac{\partial u_0}{\partial x}\cos\theta + \frac{\partial u_0}{\partial y}\sin\theta\right) \\ \text{on } \partial\tilde{\Omega} : r = r^+ = \sqrt{\frac{2\sqrt{3}}{\pi}}b; \end{gathered} \tag{9.12}$$

$$\tilde{u}_1 \to 0; \quad \frac{\partial \tilde{u}_1}{\partial r} \to 0 \quad \text{for } r \to \infty. \tag{9.13}$$

The boundary value problem (9.9)–(9.13) has the following solution:

$$u_1^- = A_1 r\cos\theta + A_2 r\sin\theta;$$
$$u_1^+ = \left(B_1 r + \frac{C_1}{r}\right)\cos\theta + \left(B_2 r + \frac{C_2}{r}\right)\sin\theta; \tag{9.14}$$
$$\tilde{u}_1 = \frac{D_1}{r}\cos\theta + \frac{D_2}{r}\sin\theta,$$

where $A_1,\ A_2,\ B_1,\ B_2,\ C_1,\ C_2,\ D_1,\ D_2$ are arbitrary constants.

Note that expression for u_1^- in Eq. (9.14) is written on account of boundedness of the function u_1^- and its derivative $\frac{\partial u_1^-}{\partial r}$ at $r = 0$, and that expression for $\tilde{u}_1$ satisfies the decay conditions (9.13) for $r \to \infty$.

The solution (9.14) contains eight arbitrary constants that can be determined from the relations (9.11) and (9.12). The systems for the constants $A_1,\ B_1,\ C_1,\ D_1$ and $A_2,\ B_2,\ C_2,\ D_2$ are completely similar. The solution for constants $A_1,\ B_1,\ C_1,\ D_1$ is the following:

$$A_1 = A_1^*\frac{\partial u_0}{\partial x};\quad B_1 = B_1^*\frac{\partial u_0}{\partial x};\quad C_1 = C_1^*\frac{\partial u_0}{\partial x};\quad D_1 = D_1^*\frac{\partial u_0}{\partial x},$$
$$A_1^* = -1 + 4\tilde{\lambda}b^2\Delta;\quad B_1^* = -1 + 2(\lambda+1)\tilde{\lambda}b^2\Delta;$$
$$C_1^* = -2(\lambda-1)\tilde{\lambda}a^2b^2\Delta, \tag{9.15}$$
$$D_1^* = \frac{4b^2}{\pi}\left(1 - 2\left((\lambda-1)\frac{\pi a^2}{2\sqrt{3}} + (\lambda+1)b^2\right)\Delta\right),$$

where

$$\Delta = \left((\tilde{\lambda}+1)(\lambda+1)b^2 - (\tilde{\lambda}-1)(\lambda-1)\frac{\pi a^2}{2\sqrt{3}}\right)^{-1}. \tag{9.16}$$

The solution for constants $A_2,\ B_2,\ C_2,\ D_2$ is obtained by the following replacements:

$$A_1 \to A_2;\quad B_1 \to B_2;\quad C_1 \to C_2;\quad D_1 \to D_2\left(\frac{\partial u_0}{\partial x} \to \frac{\partial u_0}{\partial y}\right). \tag{9.17}$$

Next step in the solution of the problem is derivation of homogenized relations. The homogenization should be carried out over the entire three-phase domain $\Omega^* = \Omega_i^+ \cup \Omega_i^- \cup \tilde{\Omega}$, i.e., over the matrix, inclusion and the outer homogeneous region. Therefore, the homogenization operator is generalized in this case as follows:

$$\widetilde{(\ldots)} = \frac{1}{|\Omega^*|}\left(\iint_{\Omega_i^+}(\ldots)\,d\xi d\eta + \lambda\iint_{\Omega_i^-}(\ldots)\,d\xi d\eta + \tilde{\lambda}\iint_{\tilde{\Omega}}(\ldots)\,d\xi d\eta\right), \tag{9.18}$$

where $|\Omega^*| = |\Omega_i^+ \cup \Omega_i^- \cup \tilde{\Omega}|$ is a measure of the three-phase domain the boundary of which can be formally considered as a circle of infinite radius $\tilde{b} \to \infty$.

Thus, the following equation should be homogenized:

$$\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + 2\frac{\partial^2 u_1^+}{\partial x \partial \xi} + 2\frac{\partial^2 u_1^+}{\partial y \partial \eta} + \frac{\partial^2 u_2^+}{\partial \xi^2} + \frac{\partial^2 u_2^+}{\partial \eta^2}$$
$$+ \lambda \left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + 2\frac{\partial^2 u_1^-}{\partial x \partial \xi} + 2\frac{\partial^2 u_1^-}{\partial y \partial \eta} + \frac{\partial^2 u_2^-}{\partial \xi^2} + \frac{\partial^2 u_2^-}{\partial \eta^2} \right)$$
$$+ \tilde{\lambda} \left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + 2\frac{\partial^2 \tilde{u}_1}{\partial x \partial \xi} + 2\frac{\partial^2 \tilde{u}_1}{\partial y \partial \eta} + \frac{\partial^2 \tilde{u}_2}{\partial \xi^2} + \frac{\partial^2 \tilde{u}_2}{\partial \eta^2} \right) = F. \tag{9.19}$$

Application of the homogenization operator (9.18) to Eq. (9.19) yields the following homogenized equation:

$$\frac{1}{|\Omega^*|} \left[\iint_{\Omega_i^+} \left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 u_1^+}{\partial x \partial \xi} + \frac{\partial^2 u_1^+}{\partial y \partial \eta} \right) d\xi d\eta \right.$$
$$+ \oint_{\partial \Omega_i} \left(\frac{\partial u_2^+}{\partial \bar{n}} + \frac{\partial u_1^+}{\partial n} \right) d\ell + \oint_{\partial \tilde{\Omega}} \left(\frac{\partial u_2^+}{\partial \bar{\mathbf{n}}} + \frac{\partial u_1^+}{\partial \mathbf{n}} \right) d\ell$$
$$+ \lambda \iint_{\Omega_i^-} \left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 u_1^-}{\partial x \partial \xi} + \frac{\partial^2 u_1^-}{\partial y \partial \eta} \right) d\xi d\eta$$
$$+ \lambda \oint_{\partial \Omega_i} \left(\frac{\partial u_2^-}{\partial \bar{\mathbf{n}}} + \frac{\partial u_1^-}{\partial \mathbf{n}} \right) d\ell$$
$$+ \tilde{\lambda} \iint_{\tilde{\Omega}} \left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 \tilde{u}_1}{\partial x \partial \xi} + \frac{\partial^2 \tilde{u}_1}{\partial y \partial \eta} \right) d\xi d\eta$$
$$\left. + \tilde{\lambda} \oint_{\partial \tilde{\Omega}} \left(\frac{\partial \tilde{u}_2}{\partial \bar{\mathbf{n}}} + \frac{\partial \tilde{u}_1}{\partial \mathbf{n}} \right) d\ell + \tilde{\lambda} \oint_{\partial \Omega^*} \left(\frac{\partial \tilde{u}_2}{\partial \bar{\mathbf{n}}} + \frac{\partial \tilde{u}_1}{\partial \mathbf{n}} \right) d\ell \right] = F. \tag{9.20}$$

The homogenized equation (9.20) can be transformed as follows on account of the expressions (9.14)–(9.17) for $u_1^+,\ u_1^-,\ \tilde{u}_1$:

$$\frac{1}{|\Omega^*|} [|\Omega_i^+|(B_1^* + 1) + \lambda |\Omega_i^-|(A_1^* + 1) + \tilde{\lambda}|\tilde{\Omega}|] \Delta u_0 = F, \tag{9.21}$$

where q is the effective coefficient:

$$q = \frac{1}{|\Omega^*|} [|\Omega_i^+|(B_1^* + 1) + \lambda |\Omega_i^-|(A_1^* + 1) + \tilde{\lambda}|\tilde{\Omega}|]. \tag{9.22}$$

The effective coefficient q defined by Eq. (9.22) is the same as the effective parameter $\tilde{\lambda}$ used in the TPhM to describe the equivalent homogeneous medium $\tilde{\Omega}$. Therefore, equating $\tilde{\lambda}$ to the expression for q in Eq. (9.22) yields the linear algebraic equation for the unknown effective coefficient $q = \tilde{\lambda}$.

The solution of this equation yields the following analytical expression for the sought effective coefficient of thermal conductivity:

$$q_{\mathrm{TPhM}} = \tilde{\lambda} = \frac{1 - \frac{\pi a^2}{2\sqrt{3}b^2} + \lambda\left(1 + \frac{\pi a^2}{2\sqrt{3}b^2}\right)}{1 + \frac{\pi a^2}{2\sqrt{3}b^2} + \lambda\left(1 - \frac{\pi a^2}{2\sqrt{3}b^2}\right)}. \tag{9.23}$$

Note that the expression (9.23) derived using the TPhM coincides with the formula obtained in Ref. 14. And it represents the upper Hashin–Shtrikman bound if $0 \leq \lambda \leq 1$, and the lower Hashin–Shtrikman bound if $1 \leq \lambda < \infty$.

10. Asymptotic Analysis of Solutions for Inclusions of Small Sizes

The expression (9.23) is obtained using TPhM on account of the assumption of small sizes of inclusions. From the other side, the relations (6.1), (6.2), (7.3), and (8.4), derived using the lubrication theory yield effective characteristics of fiber-reinforced composites of hexagonal structure in case of medium-sized inclusions. In this connection it is interesting to compare the results of the above two approaches and to assess the areas of their applicability.

The TPhM solution (9.23) for the composites with extremely large conductivity of fibers ($\lambda \to \infty$) yields (using the adopted normalization $b = 1$):

$$q_{\mathrm{TPhM}} = \frac{1 + \frac{\pi a^2}{2\sqrt{3}}}{1 - \frac{\pi a^2}{2\sqrt{3}}}. \tag{10.1}$$

The expansion of Eq. (10.1) in powers of a for the small sizes of inclusions ($a \to 0$) yields:

$$q_{\mathrm{TPhM}} = \frac{1 + \frac{\pi a^2}{2\sqrt{3}}}{1 - \frac{\pi a^2}{2\sqrt{3}}} = 1 + \frac{\pi a^2}{\sqrt{3}} + \frac{\pi^2 a^4}{6} + O(a^6). \tag{10.2}$$

Equation (10.2) coincides with the expansion of Eq. (6.1) up to terms of order a^2 inclusive:

$$\begin{aligned} q_{\mathrm{Med.Incl.}} &= 1 + \frac{\sqrt{3}a^2}{\sqrt{1-a^2}} \arctan\frac{1}{\sqrt{3(1-a^2)}} \\ &\quad + \frac{a^2}{\sqrt{3}}\left[\frac{2}{\sqrt{1-a^2}}\left(2\arctan\frac{1-a}{\sqrt{1-a^2}} + \arctan\frac{a}{\sqrt{1-a^2}}\right) - \frac{\pi}{2}\right] \\ &= 1 + \frac{\pi a^2}{\sqrt{3}} + \frac{3 + 2\pi\sqrt{3}}{8}a^4 + O(a^6). \end{aligned} \tag{10.3}$$

A similar coincidence takes place in the case of extremely small conductivity of inclusions ($\lambda \to 0$) and their small sizes ($a \to 0$). Indeed, Eqs. (7.3) and (9.23) yield

$$q_{\text{TPhM}} = \frac{1 - \frac{\pi a^2}{2\sqrt{3}}}{1 + \frac{\pi a^2}{2\sqrt{3}}} = 1 - \frac{\pi a^2}{\sqrt{3}} + \frac{\pi^2 a^4}{6} + O(a^6); \tag{10.4}$$

$$\begin{aligned} q_{\text{Med.Incl.}} &= 1 + \frac{\sqrt{3}a^2}{\sqrt{1-a^2}} \arctan \frac{1}{\sqrt{3(1-a^2)}} \\ &\quad + \frac{a^2}{\sqrt{3}} \left[\frac{2}{\sqrt{1-a^2}} \left(2 \arctan \frac{1-a}{\sqrt{1-a^2}} + \arctan \frac{a}{\sqrt{1-a^2}} \right) - \frac{\pi}{2} \right] \Bigg\}^{-1} \\ &= 1 - \frac{\pi a^2}{\sqrt{3}} + \frac{8\pi^2 - 6\pi\sqrt{3} - 9}{24} a^4 + O(a^6). \end{aligned} \tag{10.5}$$

The solution obtained using TPhM for arbitrary finite values of conductivity of inclusions λ and their small sizes ($a \to 0$) yields the following expansion:

$$\begin{aligned} q_{\text{TPhM}} &= \frac{1 - \frac{\pi a^2}{2\sqrt{3}} + \lambda\left(1 + \frac{\pi a^2}{2\sqrt{3}}\right)}{1 + \frac{\pi a^2}{2\sqrt{3}} + \lambda\left(1 - \frac{\pi a^2}{2\sqrt{3}}\right)} \\ &= 1 + \frac{\lambda - 1}{\lambda + 1} \frac{\pi a^2}{\sqrt{3}} + \frac{(\lambda-1)^2}{(\lambda+1)^2} \frac{\pi^2 a^4}{6} + O(a^6) \\ &\approx 1 + 1.814 \frac{\lambda - 1}{\lambda + 1} a^2 + 1.645 \left(\frac{\lambda - 1}{\lambda + 1} \right)^2 a^4 + O(a^6), \end{aligned} \tag{10.6}$$

coinciding with an accuracy of order a^2 inclusive with the expansions of Eqs. (6.2) and (8.4), which describe the effective coefficients found in accordance with the lubrication theory for medium-sized inclusions:

$$\begin{aligned} q_{\text{Med.Incl.}}|_{\lambda>1} &= 1 + \frac{\sqrt{3}\gamma a^2}{\sqrt{1-\gamma a^2}} \arctan \frac{\sqrt{3}}{3\sqrt{1-\gamma a^2}} \\ &\quad + \frac{\sqrt{3}\gamma a^2}{3} \left[\frac{2}{\sqrt{1-\gamma a^2}} \left(2 \arctan \frac{\sqrt{1-\sqrt{\gamma} a}}{\sqrt{1+\sqrt{\gamma} a}} \right.\right. \\ &\quad \left.\left. + \arctan \left(\frac{\sqrt{\gamma} a}{\sqrt{1-\gamma a^2}} \right) \right) - \frac{\pi}{2} \right] \\ &= 1 + \frac{\lambda - 1}{\lambda + 1} \frac{\pi a^2}{\sqrt{3}} + \frac{(\lambda-1)^2}{(\lambda+1)^2} \frac{(3 + 2\pi\sqrt{3}) a^4}{8} + O(a^6) \\ &\approx 1 + 1.814 \frac{\lambda - 1}{\lambda + 1} a^2 + 1.735 \left(\frac{\lambda - 1}{\lambda + 1} \right)^2 a^4 + O(a^6); \end{aligned} \tag{10.7}$$

$$q_{\text{Med.Incl.}}|_{\lambda<1} = \left\{1 - \frac{\sqrt{3}\gamma a^2}{\sqrt{1+\gamma a^2}} \arctan \frac{\sqrt{3}}{3\sqrt{1+\gamma a^2}} - \frac{\sqrt{3}\gamma a^2}{3}\left[\frac{2}{\sqrt{1+\gamma a^2}}\left(2\arctan\frac{\sqrt{1-\sqrt{-\gamma}a}}{\sqrt{1+\sqrt{-\gamma}a}} + \arctan\left(\frac{\sqrt{-\gamma}a}{\sqrt{1+\gamma a^2}}\right)\right) - \frac{\pi}{2}\right]\right\}^{-1}$$

$$= 1 + \frac{\lambda-1}{\lambda+1}\frac{\pi a^2}{\sqrt{3}} + \frac{(\lambda-1)^2}{(\lambda+1)^2}\frac{(8\pi^2 - 6\pi\sqrt{3} - 9)a^4}{24} + O(a^6)$$

$$\approx 1 + 1.814\frac{\lambda-1}{\lambda+1}a^2 + 1.555\left(\frac{\lambda-1}{\lambda+1}\right)^2 a^4 + O(a^6). \tag{10.8}$$

In addition, the expansions (10.6)–(10.8) coincide up to terms of order a^2 inclusive with the expansion of the approximate analytical solution q_P obtained in Ref. 13 in the case of finite λ and non-contacting inclusions, by truncating an infinite system of equations:

$$q_P = 1 + \frac{2\gamma c}{1 - \gamma c - \frac{0.075422\gamma^2 c^6}{1-1.060283\gamma^2 c^{12}} - 0.000076\gamma^2 c^{12}}$$

$$\approx 1 + 1.814\frac{\lambda-1}{\lambda+1}a^2 + 1.645\left(\frac{\lambda-1}{\lambda+1}\right)^2 a^4 + O(a^6), \tag{10.9}$$

where $c = \frac{|\Omega_i^-|}{|\Omega_i^*|} = \frac{\pi a^2}{2\sqrt{3}}$ is volume fraction of inclusions.

In case of close values of conductivities of inclusions and matrix ($\lambda \sim 1$), the expansions of solutions (6.2), (8.4) and (9.23) given by Eqs. (10.10)–(10.12) coincide with the expansion of solution q_P given by Eq. (10.13) up to the terms of order $(\lambda-1)^1$ inclusive for any size of inclusions a:

$$q_{\text{TPhM}} = \frac{1 - \frac{\pi a^2}{2\sqrt{3}} + \lambda\left(1 + \frac{\pi a^2}{2\sqrt{3}}\right)}{1 + \frac{\pi a^2}{2\sqrt{3}} + \lambda\left(1 - \frac{\pi a^2}{2\sqrt{3}}\right)} = 1 + \frac{\pi a^2}{2\sqrt{3}}(\lambda - 1) - \frac{\pi a^2}{4\sqrt{3}}\left(1 - \frac{\pi a^2}{2\sqrt{3}}\right)(\lambda-1)^2 + O((\lambda-1)^3)$$

$$\approx 1 + 0.907a^2(\lambda-1) - 0.453a^2(1 - 0.907a^2)(\lambda-1)^2 + O((\lambda-1)^3); \tag{10.10}$$

$$q_{\text{Med.Incl.}}|_{\lambda>1} = 1 + \frac{\sqrt{3}\gamma a^2}{\sqrt{1-\gamma a^2}}\arctan\frac{\sqrt{3}}{3\sqrt{1-\gamma a^2}}$$

$$+\frac{\sqrt{3}\gamma a^2}{3}\left[\frac{2}{\sqrt{1-\gamma a^2}}\left(2\arctan\frac{\sqrt{1-\sqrt{\gamma}a}}{\sqrt{1+\sqrt{\gamma}a}}\right.\right.$$

$$\left.\left.+\arctan\left(\frac{\sqrt{\gamma}a}{\sqrt{1-\gamma a^2}}\right)\right)-\frac{\pi}{2}\right]$$

$$=1+\frac{\pi a^2}{2\sqrt{3}}(\lambda-1)-\frac{\pi a^2}{4\sqrt{3}}\left(1-\frac{3a^2}{4}\left(1+\frac{\sqrt{3}}{2\pi}\right)\right)$$

$$\times(\lambda-1)^2+O((\lambda-1)^3)\approx 1+0.907a^2(\lambda-1)$$

$$-0.453a^2(1-0.957a^2)(\lambda-1)^2+O((\lambda-1)^3); \tag{10.11}$$

$$q_{\text{Med.Incl.}}|_{\lambda<1}=\left\{1-\frac{\sqrt{3}\gamma a^2}{\sqrt{1+\gamma a^2}}\arctan\frac{\sqrt{3}}{3\sqrt{1+\gamma a^2}}\right.$$

$$-\frac{\sqrt{3}\gamma a^2}{3}\left[\frac{2}{\sqrt{1+\gamma a^2}}\left(2\arctan\frac{\sqrt{1-\sqrt{-\gamma}a}}{\sqrt{1+\sqrt{-\gamma}a}}\right.\right.$$

$$\left.\left.\left.+\arctan\left(\frac{\sqrt{-\gamma}a}{\sqrt{1+\gamma a^2}}\right)\right)-\frac{\pi}{2}\right]\right\}^{-1}$$

$$=1+\frac{\pi a^2}{2\sqrt{3}}(\lambda-1)-\frac{\pi a^2}{4\sqrt{3}}\left(1+\frac{3a^2}{4}\left(1+\frac{\sqrt{3}}{2\pi}-\frac{4\pi}{3\sqrt{3}}\right)\right)$$

$$\times(\lambda-1)^2+O((\lambda-1)^3)\approx 1+0.907a^2(\lambda-1)$$

$$-0.453a^2(1-0.857a^2)(\lambda-1)^2+O((\lambda-1)^3); \tag{10.12}$$

$$q_P=1+\frac{2\gamma c}{1-\gamma c-\frac{0.075422\gamma^2c^6}{1-1.060283\gamma^2c^{12}}-0.000076\gamma^2c^{12}}$$

$$\approx 1+0.907a^2(\lambda-1)-0.453a^2(1-0.907a^2)(\lambda-1)^2$$

$$+O((\lambda-1)^3). \tag{10.13}$$

11. Numerical Results

In this section, the above obtained solutions are analyzed and compared with the known data in some particular cases.

The expressions (10.1), (6.1), (3.9) and (3.10) define the asymptotic behavior of the effective coefficient of thermal conductivity of fiber-reinforced composite material of hexagonal structure in the case of very large conductivity of inclusions ($\lambda \to \infty$) and with inclusions of small, medium, large and close to the limit ($a \to 1$) sizes respectively.

The plots of the effective coefficients calculated using Eqs. (10.1), (6.1), (3.9) and (3.10), in comparison with the known asymptotic result (see Ref. 18); and numerical and analytical data (see Ref. 13) are shown in Figs. 10 and 11.

The dependence of the effective coefficient of thermal conductivity on the size of inclusions a in the case of large but finite conductivity of inclusions is shown in

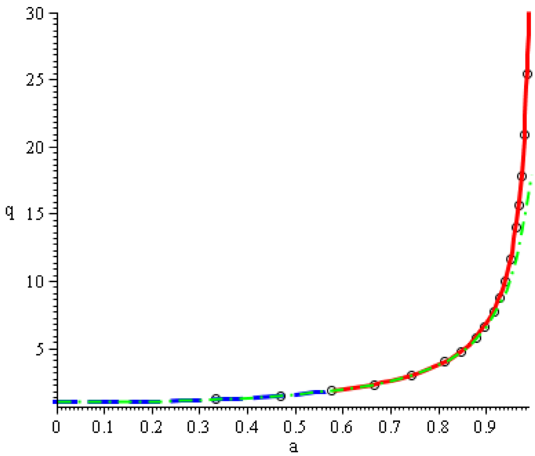

Fig. 10. Effective coefficient of thermal conductivity of fiber-reinforced composite material of hexagonal structure with inclusions of small ($a \to 0$), medium ($a \ll 1$) and large ($0 \ll a < 1$) sizes, and with very large conductivity ($\lambda \to \infty$).

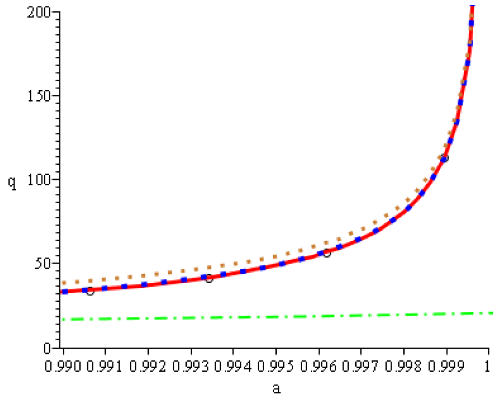

Fig. 11. Effective coefficient of thermal conductivity of fiber-reinforced composite material of hexagonal structure with inclusions of very large size ($a \to 1$), and very large conductivity ($\lambda \to \infty$).

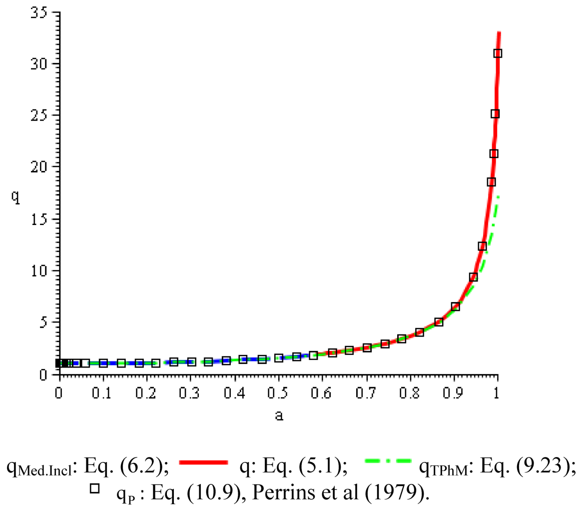

Fig. 12. Effective coefficient of thermal conductivity of fiber-reinforced composite material of hexagonal structure with inclusions of arbitrary sizes ($0 \le a \le 1$), and large conductivity ($\lambda = 100$).

Fig. 12, where the value for $\lambda = 100$ is assumed as an example. Equations (5.1), (6.2) and (9.23) are used as defining expressions, and compared with the approximate analytical formula (10.9) from Ref. 13.

Effective coefficients of thermal conductivity calculated according to Eqs. (5.1), (6.2), (8.2), (8.4) and (9.23) are plotted in Figs. 13 and 14 for different sizes of

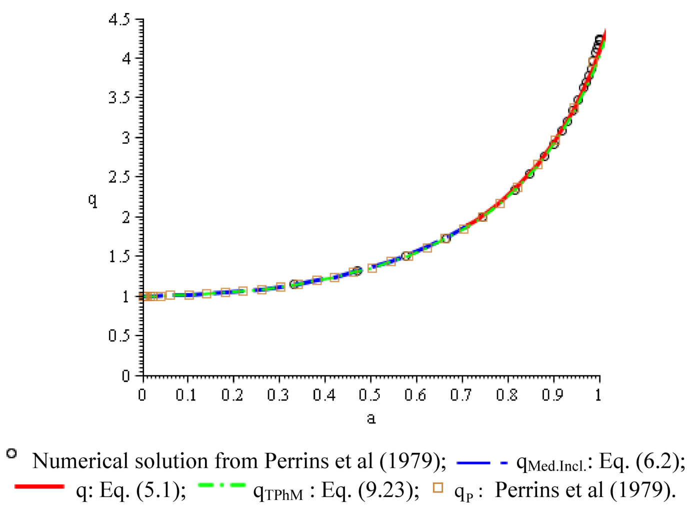

Fig. 13. Effective coefficient of thermal conductivity of fiber-reinforced composite material of hexagonal structure with inclusions of arbitrary sizes ($0 \le a \le 1$), and conductivities of inclusions and matrix of the same order $\lambda = 5$.

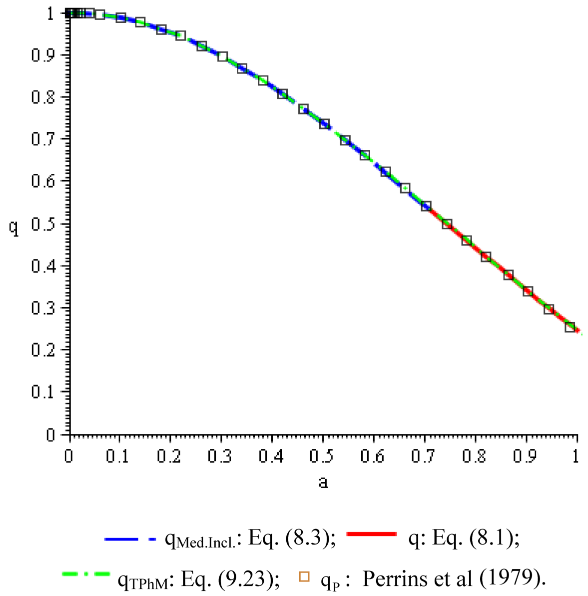

Fig. 14. Effective coefficient of thermal conductivity of fiber-reinforced composite material of hexagonal structure with inclusions of arbitrary sizes ($0 \leq a \leq 1$), and conductivities of inclusions and matrix of the same order $\lambda = 0.2$.

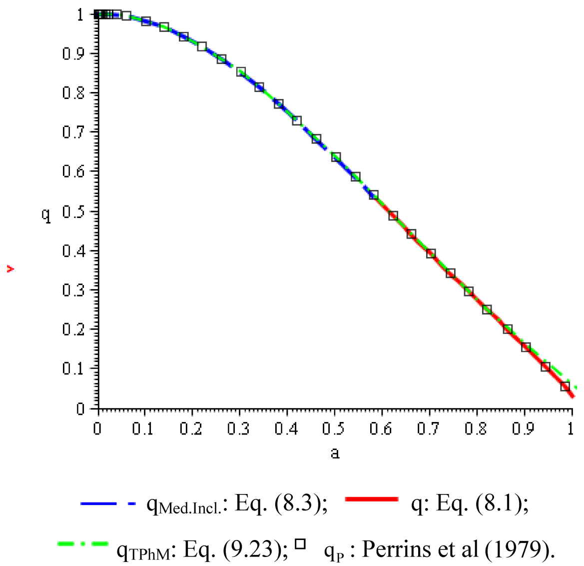

Fig. 15. Effective coefficient of thermal conductivity of fiber-reinforced composite material of hexagonal structure with inclusions of arbitrary sizes ($0 \leq a \leq 1$), and small conductivity of inclusions $\lambda = 0.01$.

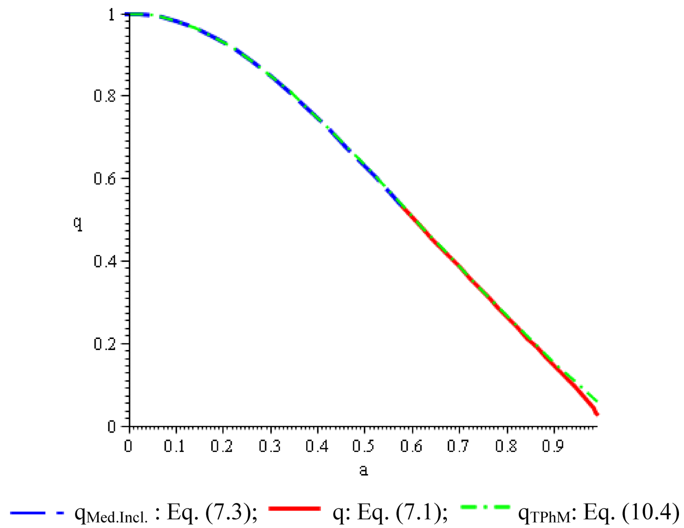

Fig. 16. Effective coefficient of thermal conductivity of fiber-reinforced composite material of hexagonal structure with inclusions of small ($a \to 0$), medium ($a \ll 1$) and large ($0 \leq a \leq 1$) sizes, and extremely small conductivity of inclusions ($\lambda \to 0$).

inclusions in the case of the conductivities of inclusions and matrix of the same order ($\lambda \sim 1$).

Effective coefficients of thermal conductivity calculated according to Eqs. (8.2), (8.4) and (9.23) are plotted in Fig. 15 for different sizes of inclusions and small conductivity of inclusions: $\lambda = 0.01$.

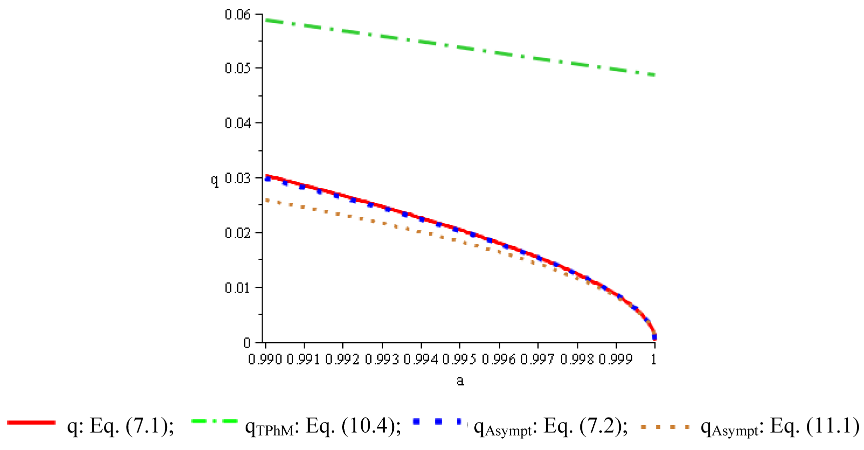

Fig. 17. Effective coefficient of thermal conductivity of fiber-reinforced composite material of hexagonal structure with inclusions of large sizes ($a \to 1$), and extremely small conductivity of inclusions ($\lambda \to 0$).

Figures 16 and 17 show plots of the effective coefficients of thermal conductivity calculated using Eqs. (7.1)–(7.3) and (10.4), and compared with the asymptotic formula from Ref. 18 which can be written as follows on account of Eqs. (3.11) and (8.1) for $\lambda \to 0$:

$$q_{\text{Asympt}}|_{\lambda\to 0} = \frac{\sqrt{1-a^2}}{\sqrt{3}\pi}. \tag{11.1}$$

12. Summary and Discussion

In this paper, a fiber-reinforced composite material with periodically arranged hexagonal array of circular cylindrical fibers is analyzed. The analytical governing relations for the problem of thermal conductivity for such composite material are derived using the asymptotic homogenization method. The relevant unit cell problems are obtained and solved in several practically important cases. These solutions are essential for deriving the formulae for the effective properties of the composite material.

The lubrication theory is applied to obtain the asymptotic solution of the unit cell problems in the cases of inclusions of large and close to the limiting sizes ($a \to 1$) and in cases of very large conductivity of inclusions ($\lambda \to \infty$). The lubrication theory approach is based on replacing the boundary value problem in the original domain with a problem formulated in the domain with more simple geometry. In determining the effective characteristics of the composite material the hexagonal shape of the unit cell is taken into account by transforming the geometrical parameter of unit cell into the variable function of the coordinates.

Using the lubrication theory the asymptotic representation of the effective coefficient of thermal conductivity given by Eq. (3.10) is obtained for the very high conductivity of inclusions ($\lambda \to \infty$) and very large sizes of inclusions ($a \to 1$). The principal term of the asymptotic expansion (3.10) coincides with the asymptotic formula obtained in Ref. 18.

The lubrication theory is generalized and the analytical expression for the effective coefficient of thermal conductivity given by Eqs. (5.1), (5.2) is derived in the case of finite values of conductivity of inclusions ($\lambda \gg 1$) and their large sizes:

$$\sqrt{\frac{\lambda-1}{\lambda+1}}a \geq \frac{1}{\sqrt{3}}.$$

The lubrication theory is further generalized for the inclusions of medium sizes. Depending on the physical characteristics of the inclusions the effective coefficient is described by Eq. (6.1) for $\lambda \to \infty$ and $a \leq \frac{1}{\sqrt{3}}$, and by Eq. (6.2) for $\lambda \gg 1$ and $\sqrt{\frac{\lambda-1}{\lambda+1}}a \leq \frac{1}{\sqrt{3}}$, respectively.

The asymptotic expressions for the effective coefficients of thermal conductivity are derived on the basis of the obtained solutions and Keller's theorem in the cases of very small conductivity of inclusions ($\lambda \to 0$) and large sizes of inclusions, namely: Eq. (7.1) for $a \geq \frac{1}{\sqrt{3}}$; Eq. (7.2) for $a \to 1$; and Eq. (7.3) for $a \leq \frac{1}{\sqrt{3}}$.

The generalized analytical expressions for the effective coefficients of the hexagonal composite structure are obtained in the case of small conductivity of inclusions ($\lambda \ll 1$), namely Eqs. (8.4), (8.2) and (8.3) for medium, large and very large inclusions, respectively.

The TPhM was used to analyze the variations of the effective properties of the composite with decrease in the sizes of inclusions. In this approach, the entire composite material, except one unit cell, is replaced by a homogeneous medium with the unknown and sought effective coefficient. For the hexagonal composite structure the obtained effective coefficient given by Eq. (9.23) coincides with the upper Hashin–Shtrikman bound if $0 \leq \lambda \leq 1$, and the lower Hashin–Shtrikman bound if $1 \leq \lambda < \infty$.

The asymptotic analysis of the obtained solutions for the small-sized inclusions was performed. It proved that the results of lubrication theory for medium-sized inclusions, namely Eqs. (6.1), (6.2), (7.3), (8.4) can be extended to the small-sized inclusions:

(i) expansion of these relations in series for $a \to 0$ is the same up to the terms of the order a^2 inclusive for all the values $0 \leq \lambda < \infty, \lambda \to \infty$ as the corresponding solution given by the Eq. (9.23), obtained using the TPhM, and the approximate solution given by Eq. (10.9) obtained in Ref. 13;
(ii) expansion of the expressions (6.2), (8.4) in series for $\lambda \to 1$ is the same up to the terms of the order $(\lambda - 1)^1$ inclusive for all values of a as the expansions of solution (9.23) obtained using the TPhM, and the approximate solution (10.9) obtained in Ref. 13.

The obtained analytical results are illustrated graphically and compared with the known numerical and asymptotic data in some particular cases, and a very good agreement is demonstrated. The numerical analysis and the presented plots proved that:

(i) both, the lubrication theory and the TPhM provide practically identical results for all sizes of inclusions $0 \leq a \leq 1$ (small, medium, large and very large) if the conductivities of inclusions and matrix are close ($\lambda \sim 1$), and
(ii) these results are in a good agreement with the approximate analytical formula (10.9) and the numerical data from Ref. 13.
(iii) the variation of the effective coefficient is identical for large but finite values of conductivity of inclusions ($\lambda \gg 1$) and the small but nonzero conductivity of inclusions ($0 < \lambda \ll 1$);
(iv) Figures 11 and 15 show that results for the effective coefficient of thermal conductivity calculated from the asymptotic formula (11.1) from Ref. 18, differs by 10–15% at values $a \approx 0.99$ from our more rigorous formulas (7.1) and (7.2), but they becomes practically identical at $a \to 1$;
(v) the TPhM works well for small and medium-sized inclusions; while increasing their sizes to large and very large leads to undervalued results if $\lambda \gg 1$,

and overvalued results if $0 < \lambda \ll 1$. In this case, the TPhM results can be considered as effective estimates for the effective coefficient corresponding to the Hashin–Shtrikman bounds;

(vi) solution obtained using the lubrication theory adequately describes the effective coefficient in the whole range of sizes of inclusions;

(vii) it is shown in the limiting cases of conductivities of inclusions ($\lambda \to \infty$ and $\lambda \to 0$) that the TPhM solution (9.23) provides correct estimation of the effective coefficient for small and medium-sized inclusions, and gives an estimate for large-sized inclusions; but solution (9.23) is not accurate in the limiting case $a \to 1$. The lubrication theory solution provides correct estimation of the effective coefficient of the hexagonal composite in the entire range of sizes of inclusions: $0 \leq a < 1$ and $a \to 1$, including its asymptotic behavior in the latter case.

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