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Effects of randomness on chaos and order of coupled logistic maps

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Abstract

Natural systems are essentially nonlinear being neither completely ordered nor completely random. These nonlinearities are responsible for a great variety of possibilities that includes chaos. On this basis, the effect of randomness on chaos and order of nonlinear dynamical systems is an important feature to be understood. This Letter considers randomness as fluctuations and uncertainties due to noise and investigates its influence in the nonlinear dynamical behavior of coupled logistic maps. The noise effect is included by adding random variations either to parameters or to state variables. Besides, the coupling uncertainty is investigated by assuming tinny values for the connection parameters, representing the idea that all Nature is, in some sense, weakly connected. Results from numerical simulations show situations where noise alters the system nonlinear dynamics.

Keywords: Chaos; Nonlinear dynamics; Logistic map; Noise; Randomness; Synchronization; Complex systems

1. Introduction

The term complexity has been used to denote the main characteristics related to complex system behavior associated with complicated and intricate features. The detailed comprehension of complex system behavior is not well-established, however, there are some characteristics often exhibited by this kind of behavior. The entire system may be split into parts that are connected by intricate manners. Besides, there exist multi-scale aspects, exhibiting complex patterns [1,2]. These complexity characteristics are usually coined as emergence, self-organization, synergetics, collective behaviors, and other equivalent jargons. Chua [3] argue that local activity is the origin of complexity, explaining that all complex properties are manifestations of this new principle. Moreover, this argue says that most complex phenomena emerge to a subset of locally-active region, called the edge of chaos [4].

Natural systems have nonlinear characteristics responsible for a great variety of possibilities. Chaos is one of these possibilities that has an intrinsically richness related to its structure. Because of that, there are benefits for natural systems of adopting chaotic regimes with their wide range of potential behaviors. Besides, chaos is related to long-term unpredictability and may be geometrically understood considering a sequence of contraction–expansion–folder transformation, known as Smale horseshoe [5,6]. In the past, most of contributions related to chaotic dynamics were concentrated on the time evolution analysis of low-dimensional dynamical systems. Nevertheless, several natural systems must be investigated according to a high-dimensional approach. Recently, the spatiotemporal chaos has attracted so much attention due to its theoretical and practical applications [1,7–11].

Poon and Grebogi [2] argue that natural systems are neither completely ordered nor completely random, and therefore, the complex behavior has both elements of order and randomness. On this basis, it is an important feature to understand if randomness is a fundamental principle governing natural systems or if it is a limitation in comprehending complex systems [12]. Besides, it is important to evaluate the randomness influence on chaos and order of nonlinear dynamical systems. The literature presents many reports dealing with different aspects of noise in nonlinear dynamics. These articles evaluate noise-induced chaos, synchronization or control of dynamical

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systems [13–16]. Moreover, there are many studies dealing with noise robustness of techniques employed for nonlinear analysis [17–21]. Therefore, it is important to argue the relationship between chaos and complexity and also between chaos and noise [12,22].

The chaos study origin was characterized by the investigation of simple problems with very complicated dynamics. An emblematic example is the logistic map applied in biological, economic and social sciences [23]. In this article, this "simplicity" is exploited in order to investigate the effect of randomness, represented by fluctuations and uncertainties due to noise. Coupled logistic maps, which study has been motivated by the description of spatial heterogeneity on population dynamics, are used with this aim [1,7,24–28]. The effect of noise in the nonlinear dynamical behavior of logistic maps are treated in some references with different objectives [14,15,29–31].

This Letter considers three kinds of situations related to randomness. Fluctuations are represented by adding random noise either to parameters or to state variables. Moreover, uncertainties are investigated by assuming tinny values for the connection parameters, representing that all Nature is, in some sense, weakly connected. Numerical simulations are carried out investigating these randomness effects in the system nonlinear dynamics.

2. Coupled logistic map

Logistic map is a simple first-order difference equation originally proposed to describe population dynamics:

$$X_{n+1} = F(X_n; \alpha) = \alpha X_n(1 - X_n). \tag{1}$$

The nonlinear dynamics of this map is well-known being discussed in different references. In order to briefly characterize its dynamics, it is presented a bifurcation diagram in Fig. 1, together with its enlargement in a specific region. This classical diagram shows a road to chaos characterized by period doubling cascades, being noticeable periodic windows inside chaotic regions, and also crisis phenomenon.

Motivated by the description of spatial heterogeneity on population dynamics, many authors are considering different coupling forms of logistic maps [24–28]. Here, two logistic maps are coupled by the connection parameter, ε, as follows:

$$\begin{cases} X_{n+1} = F(X_n; \alpha_X) + \varepsilon[F(Y_n; \alpha_Y) - F(X_n; \alpha_X)], \\ Y_{n+1} = F(Y_n; \alpha_Y) - \varepsilon[F(Y_n; \alpha_Y) - F(X_n; \alpha_X)]. \end{cases} \tag{2}$$

Analyses are developed by considering different behaviors of each map (called X-map and Y-map) and the interactions between them. As examples of behaviors of each isolated map, the following parameters are employed during the developed analysis: $\alpha = 2.5$ (period-1), $\alpha = 3.2$ (period-2), $\alpha = 3.63$ (period-6, periodic window), $\alpha = 3.64$ (near crisis), $\alpha = 3.8$ (chaos).

The forthcoming discussion is focused on coupled logistic map. At first, parameters $\alpha_X = 3.8$ (chaos), $\alpha_Y = 2.5$ (period-1) are considered and the influence of connection parameter ε is analyzed from bifurcation diagrams presented in Fig. 2. The ε value increase tends to synchronize the map behaviors, transmitting chaos from the X-map to the Y-map. Moreover, notice that there are periodic windows within chaos. The coupled map responses for some connection parameter values are shown in Fig. 3, presenting the $X_{n+1} - Y_{n+1}$ space. When $\varepsilon = 0$, there is chaos in the X-map and a period-1 response in the Y-map and

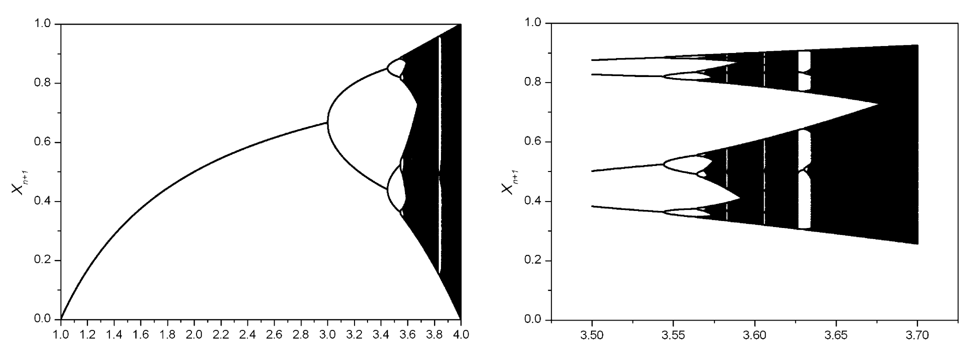

Fig. 1. Logistic map bifurcation diagram.

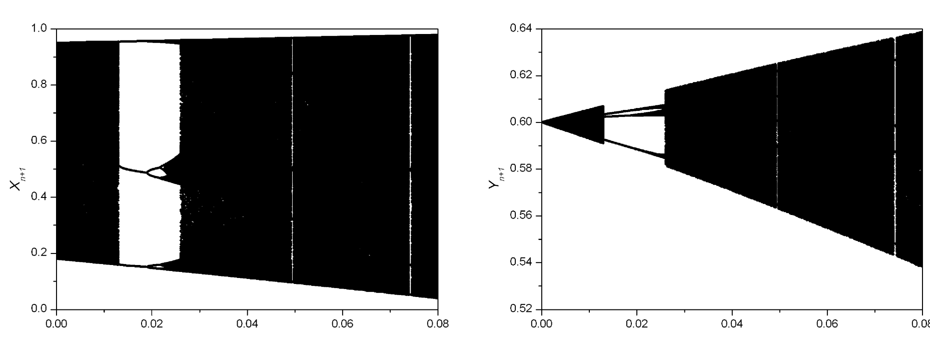

Fig. 2. Logistic map bifurcation diagram $\alpha_X = 3.8$ (chaos) and $\alpha_Y = 2.5$ (period-1).

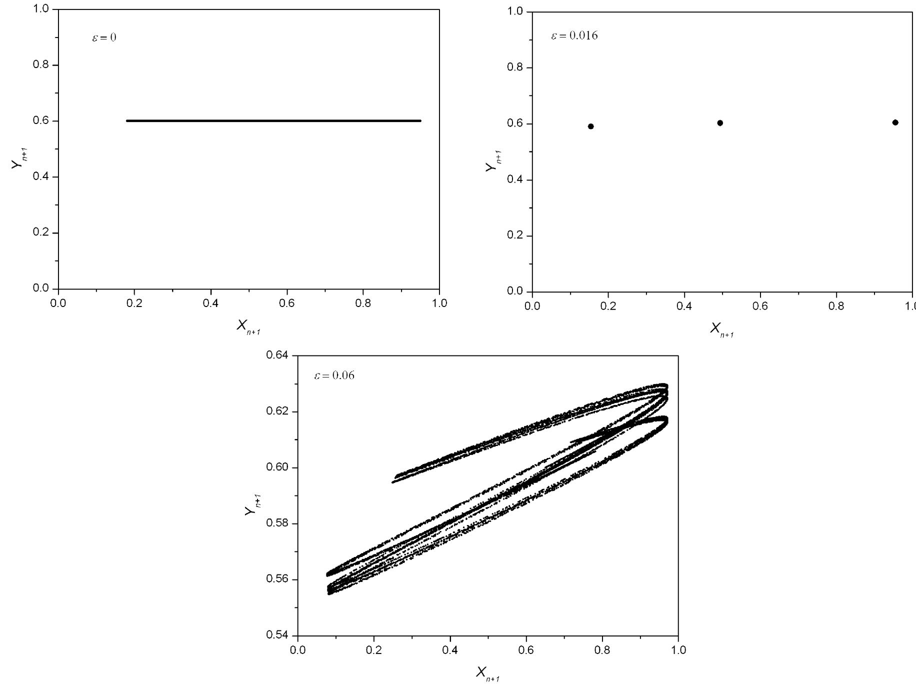

Fig. 3. Logistic map behavior for $\alpha_X = 3.8$ (chaos) and $\alpha_Y = 2.5$ (period-1) and different connections ($\varepsilon = 0$, $\varepsilon = 0.016$, $\varepsilon = 0.06$).

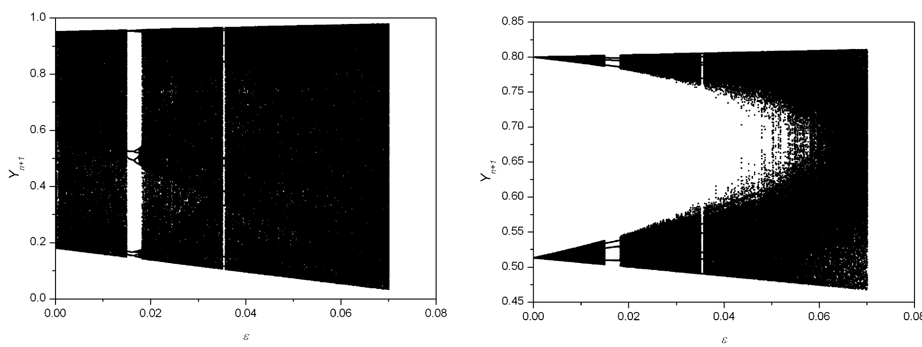

Fig. 4. Logistic map bifurcation diagram $\alpha_X = 3.8$ (chaos) and $\alpha_Y = 3.2$ (period-2).

the coupled response is represented by a horizontal line. When $\varepsilon = 0.016$, a value inside the periodic window, it is observed a period-3 coupled behavior. Chaotic behavior is observed when $\varepsilon = 0.06$.

Now, parameters are changed considering $\alpha_X = 3.8$ (chaos) and $\alpha_Y = 3.2$ (period-2). Bifurcation diagrams analyzing variations in the connection parameter, ε, is presented in Fig. 4. Once again, synchronization is observed (similar to the previous example), transmitting chaos from the X-map to the Y-map; periodic windows still existing within chaos. The coupled map responses for some values of connection parameter are presented in Fig. 5. When $\varepsilon = 0$, there is a chaos in the X-map and period-2 in the Y-map and the coupled response is represented by two horizontal lines. When $\varepsilon = 0.012$, it is observed a chaotic behavior with a disconnected attractor. By increasing connection parameter to $\varepsilon = 0.06$, there is a different chaotic attractor.

At this point, a parameter near the X-map crisis is considered ($\alpha_X = 3.64$) together with $\alpha_Y = 2.5$ (period-1). Bifurcation diagrams for this situation are presented in Fig. 6, showing a similar structure of the previous ones.

3. Fluctuations and uncertanties due to noise

Randomness influence on the coupled logistic map nonlinear dynamics is analyzed by adding random noise either to parameters or to state variables. On this basis, the following coupled map is considered:

$$\begin{cases} X_{n+1} = F(X_n; \rho_{\alpha_X}, \alpha_X) + \rho_\varepsilon \varepsilon [F(Y_n; \rho_{\alpha_Y}, \alpha_Y) \\ \qquad - F(X_n; \rho_{\alpha_X}, \alpha_X)] + (1 - \rho_X) X_n, \\ Y_{n+1} = F(Y_n; \rho_{\alpha_Y}, \alpha_Y) - \rho_\varepsilon \varepsilon [F(Y_n; \rho_{\alpha_Y}, \alpha_Y) \\ \qquad - F(X_n; \rho_{\alpha_X}, \alpha_X)] + (1 - \rho_Y) Y_n. \end{cases} \tag{3}$$

Variables ρ_{α_X}, ρ_{α_Y}, ρ_ε, ρ_X and ρ_Y are related to random numbers and their definition follow the rule: $\rho = 1 + \delta R(-1, +1)$, where $R(-1, +1)$ is a random number in the range $(-1, +1)$ and δ is the amplitude of this variation. Random numbers are generated by proper algorithms [32]. Although all variables are

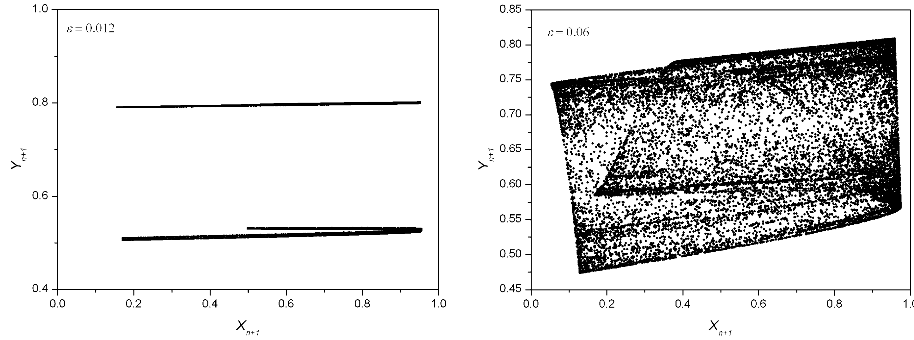

Fig. 5. Logistic map behavior for $\alpha_X = 3.8$ (chaos) and $\alpha_Y = 3.2$ (period-2) and different connections ($\varepsilon = 0.012$, $\varepsilon = 0.06$).

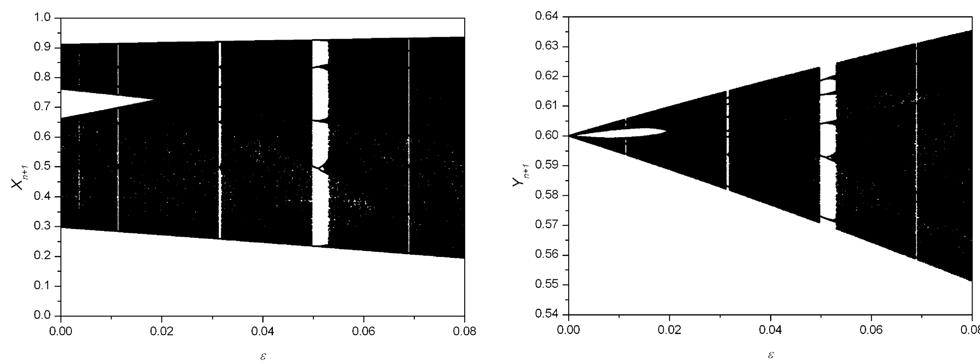

Fig. 6. Logistic map bifurcation diagram $\alpha_X = 3.64$ (near crisis) and $\alpha_Y = 2.5$ (period-1).

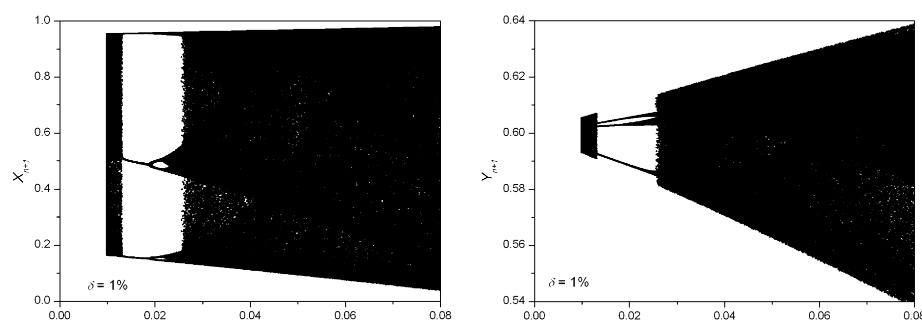

Fig. 7. Logistic map bifurcation diagram $\alpha_X = 3.8$ (chaos), $\alpha_Y = 2.5$ (period-1) and a noise $\rho_\varepsilon = \rho_\varepsilon(\delta)$, $\delta = 1\%$.

defined by the same form, in the definition of ρ_ε, however, the product $(\rho_\varepsilon \varepsilon)$ is never less than δ, which defines the smallest noise level.

The influence of fluctuations in the connection parameter is now in focus. Therefore, it is assumed $\rho_{\alpha_X} = \rho_{\alpha_Y} = \rho_X = \rho_Y = 1$ and $\rho_\varepsilon = \rho_\varepsilon(\delta)$. In order to establish a comparison with results obtained in the previous section, it is assumed $\alpha_X = 3.8$ (chaos) and $\alpha_Y = 2.5$ (period-1). By considering $\delta = 1\%$, results are analyzed from bifurcation diagrams presented in Fig. 7, which may be compared with Fig. 2. It is noticeable that the uncoupled behavior does not exist anymore since it is considered that there is always a connection due to noise. This effect implies that randomness may cause unexpected coupling. Moreover, the noise destroys some periodic windows changing some expected behaviors.

The map response coupled by the noise ($\varepsilon = 0$, $\rho_\varepsilon = \rho_\varepsilon(\delta)$) is presented in Fig. 8 for different noise levels: $\delta = 1\%$ and $\delta = 5\%$. When $\delta = 0$ (see Fig. 3), there is chaos in the X-map and a period-1 response in the Y-map, which is represented by a horizontal line in the $X_{n+1} - Y_{n+1}$ space. By increasing the noise level, δ, this horizontal line tends to become a chaotic attractor (Fig. 8). The attractor transition from the horizontal line (when $\delta = 0$—Fig. 3) to other situations where $\delta \neq 0$ (Fig. 8) suggests a multi-scale characteristic. The δ increase tends to increase the attractor region. Therefore, the randomness generates attractors related to the noise level. By comparing Fig. 3 with Fig. 8 it is possible to infer that for δ values less than 1% (and greater than 0) the chaotic attractor may be viewed as a horizontal line. Nevertheless, there exists a proper observation scale where it is possible to identify the existence of an attractor. The noise level is also related to this observation scale.

The uncertainty is now focused on considering that both maps are weakly connected ($\varepsilon = 1 \times 10^{-4}$). Philosophically speaking, this situation establishes that all Nature is, in some

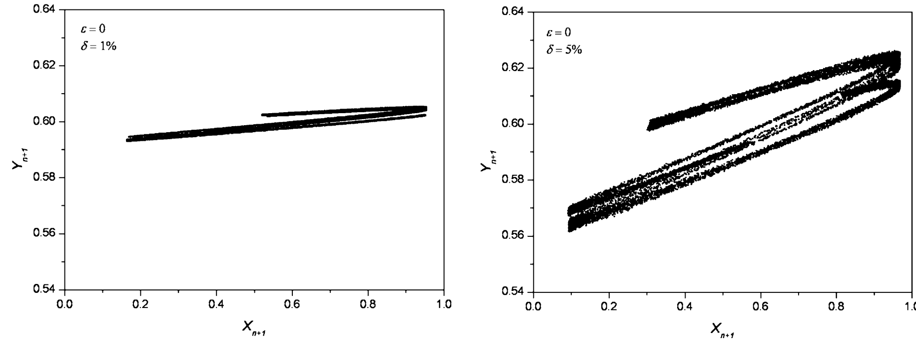

Fig. 8. Logistic map behavior for $\alpha_X = 3.8$ (chaos), $\alpha_Y = 2.5$ (period-1), $\varepsilon = 0$ and different noise levels ($\rho_\varepsilon = \rho_\varepsilon(\delta)$: $\delta = 1\%$, $\delta = 5\%$).

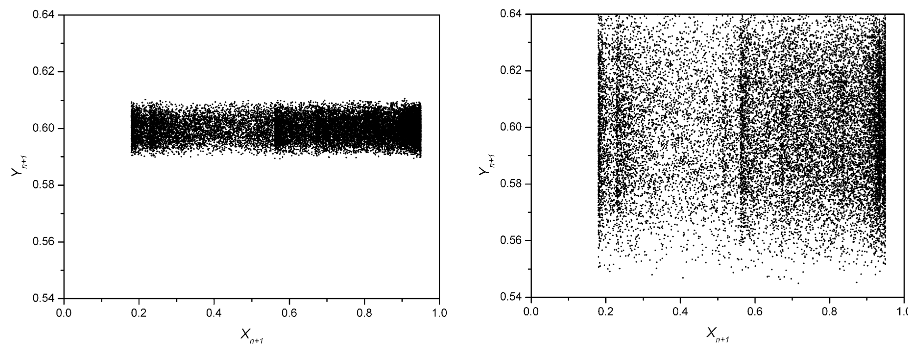

Fig. 9. Logistic map behavior for $\alpha_X = 3.8$ (chaos), $\alpha_Y = 2.5$ (period-1), $\varepsilon = 1 \times 10^{-4}$ and different noise levels ($\rho_{\alpha_Y} = \rho_{\alpha_Y}(\delta)$: $\delta = 1\%$, $\delta = 5\%$).

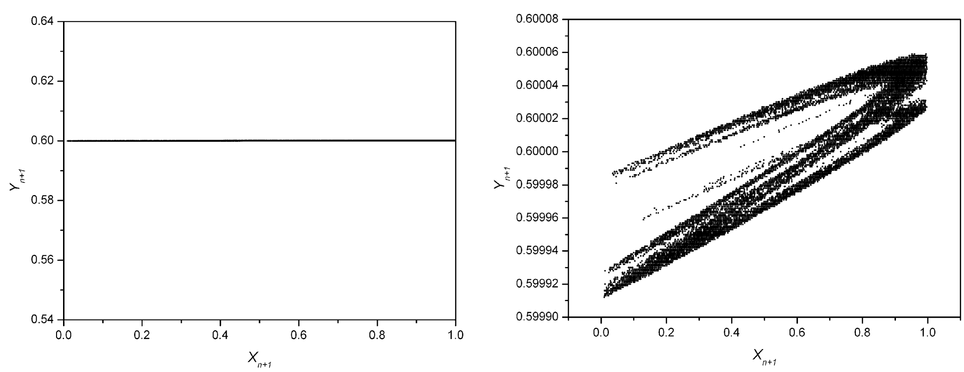

Fig. 10. Logistic map behavior for $\alpha_X = 3.8$ (chaos), $\alpha_Y = 2.5$ (period-1), $\varepsilon = 1 \times 10^{-4}$ and noise level $\rho_{\alpha_X} = \rho_{\alpha_X}(\delta)$, $\delta = 5\%$.

sense, weakly connected. These weak connections may become strong as a consequence of some events as the randomness. This situation is investigated by assuming parameter fluctuations represented by ρ_{α_X} and ρ_{α_Y}, respectively associated with the X-map and the Y-map. At first, it is considered a situation where $\alpha_X = 3.8$ (chaos) and $\alpha_Y = 2.5$ (period-1). Results related to the Y-map fluctuations (ρ_{α_Y}) are presented in Fig. 9, showing situations with different noise levels ($\delta = 1\%$ and $\delta = 5\%$). Notice that the noise level increase tends to increase the cloud of points related to randomness. Results related to the X-map fluctuations (ρ_{α_X}, $\delta = 5\%$) are presented in Fig. 10. For this case, there is a chaotic attractor that, in this scale (left side of Fig. 10), cannot be distinguished from a horizontal line. The enlargement of this response, however, shows the attractor structure (right side of Fig. 10). At this point, it should be highlighted that the Y-map noise (related to a period-1 response) has a greater influence in the system dynamics than the X-map noise (related to a chaotic response). Nevertheless, it is important to notice that for attractor characteristic observations on scales less than the noise level, the attractor appears to be a cloud of points [33].

Chaos presents sensitive dependence on initial conditions and also parameters sensitivity under certain circumstances. The crisis is a typical situation where it occurs. Therefore, it is expected that, near the crisis, noise effect becomes more effective. In order to investigate this situation, let us consider $\alpha_X = 3.64$, $\alpha_Y = 2.5$ and noise levels related to the X-map (noise $\rho_{\alpha_X} = \rho_{\alpha_X}(\delta)$, with the others equal to unit). Bifurcation diagram related to ε variation is shown in Fig. 6 for a situation without noise. Now, it is considered the response for $\varepsilon = 0$ and different noise levels (Fig. 11). For a situation without noise, the X-map presents a chaotic disconnect attractor while the Y-map presents a period-1 response. By considering noise ($\delta = 1\%$),

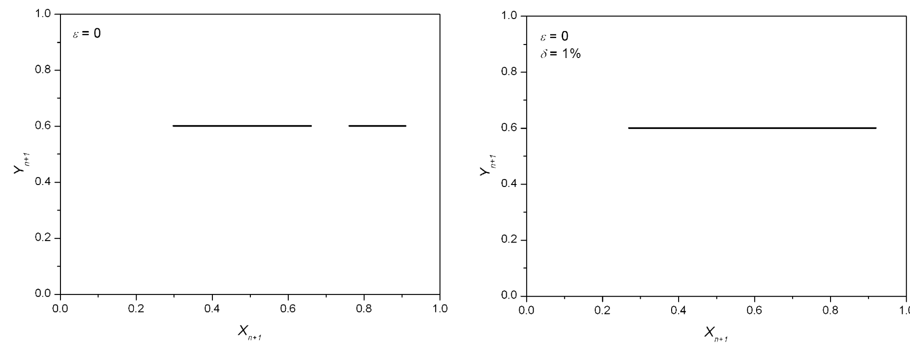

Fig. 11. Logistic map behavior for $\alpha_X = 3.64$ (near crisis), $\alpha_Y = 2.5$ (period-1), $\varepsilon = 0$ and different noise levels ($\rho_{\alpha_X} = \rho_{\alpha_X}(\delta)$: $\delta = 0$, $\delta = 1\%$).

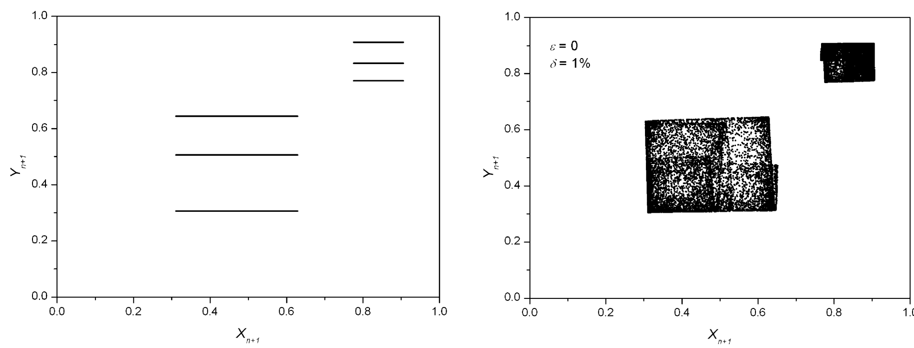

Fig. 12. Logistic map behavior for $\alpha_X = 3.64$ (near crisis), $\alpha_Y = 3.63$ (period-6, periodic window), $\varepsilon = 0$ and different noise levels ($\rho_{\alpha_X} = \rho_{\alpha_X}(\delta)$: $\delta = 0$, $\delta = 1\%$).

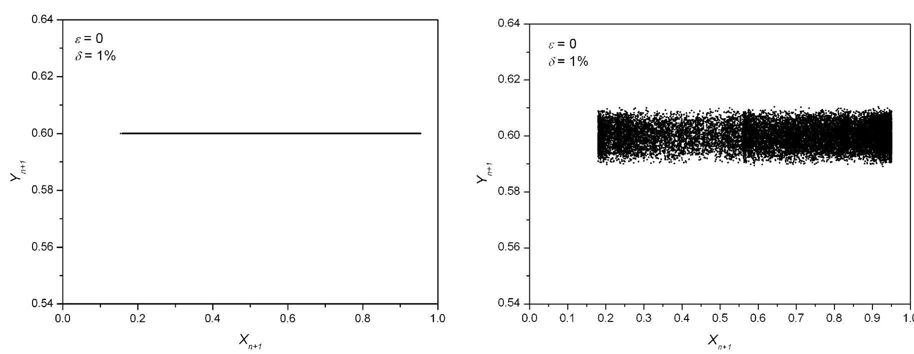

Fig. 13. Logistic map behavior for $\alpha_X = 3.8$ (chaos), $\alpha_Y = 2.5$ (period-1), $\varepsilon = 0$ and ($\rho_{\alpha_X} = \rho_{\alpha_X}(\delta)$, $\delta = 1\%$).

the attractor changes its form, highlighting the crisis phenomenon.

Dynamical responses within periodic windows are other situations where parameter sensitivity is important. In order to investigate this behavior it is considered a situation where $\alpha_X = 3.62$ (near crisis) and $\alpha_Y = 3.63$ (period-6, periodic window). Once again, it is established a comparison between situations with and without noise (Fig. 12). When $\delta = 0$ (situation without noise), the X-map presents a chaotic disconnect attractor while the Y-map presents a period-6 response. The coupled system, therefore, presents an attractor formed by six horizontal lines (Fig. 12, left side). By considering noise ($\delta = 1\%$), different structures appears (Fig. 12, right side).

The state variable noise fluctuation is now focused on. A situation where $\alpha_X = 3.8$ (chaos), $\alpha_Y = 2.5$ (period-1) and $\varepsilon = 0$ is assumed. At this point, it is assumed noise fluctuations associated with the X-map ($\rho_X = \rho_X(\delta), \delta = 1\%$) and also related to the Y-map $\rho_Y = \rho_Y(\delta)$, ($\delta = 1\%$). The left side of Fig. 13 presents the response with the X-fluctuations showing the same qualitative behavior of those without noise fluctuations ($\delta = 0$—see Fig. 3). On the other hand, the right side of Fig. 13 presents the response with the Y-fluctuations, showing that the point related to a period-1 response is replaced by a cloud of points which thickness is associated with the noise level. Once again, it should be highlighted that noise related to periodic response has a greater influence in the system behavior than the one associated with chaos.

4. Conclusions

This Letter discusses some aspects related to the effect of randomness on chaos and order of coupled logistic maps. Fluctuations and uncertainties are incorporated considering parameters and state variables random variations. Besides, since it is possible to consider that all Nature is, in some sense, weakly connected, it is investigated situations where connection parameters assume tinny values. The natural system intricate connections may be promoted by fluctuations and uncertainties, inducing, for example, synchronization among them. Multi-scale characteristic is also another important aspect that may be related to noise that can induce different attractors depending on noise level and also on observation scale. Sensitive dependence either on initial conditions or on parameters may be highly influenced by noise. Concerning the noise influence on nonlinear dynamical responses, results show that fluctuations related to periodic response has a greater influence than fluctuations related to chaotic behavior. These results may be a simple and useful manner for the comprehension of many aspects related to complex systems where chaos, order and randomness are combined.

Acknowledgements

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