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# Chaos in a shape memory two-bar truss

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**Abstract**

The study of the structural response of two-bar trusses may be very helpful to understand some of the main stability characteristics of framed structures, as well as of flat arches and of many other physical phenomena associated with bifurcation buckling. This article is concerned with the dynamic response of a shape memory two-bar truss, which is an interesting example of a structural system that exhibits both kinematic and constitutive non-linearities. A polynomial constitutive model is assumed to describe the behavior of the shape memory bars. Free and forced responses are investigated. Numerical simulations show that the system can easily reach a chaotic response. © 2002 Published by Elsevier Science Ltd.

*Keywords:* Chaos; von Mises truss; Shape memory

## 1. Introduction

The study of the two-bar truss, also known as the von Mises truss, is important to define the main stability characteristics of framed structures as well as of flat arches and of many other physical phenomena associated with bifurcation buckling [1]. As depicted in Fig. 1, this simple, plane, framed structure, is formed by two identical bars, both making an angle $\varphi$ with a horizontal line, and free to rotate around their supports and at the joint. If the structure's mass is assumed to be lumped at the node, and only vertical, symmetrical motions of the truss are considered, the resulting discrete dynamical system is essentially one dimensional.

Despite the deceiving simplicity of the von Mises truss, its non-linear dynamic response may exhibit a number of interesting, complex behaviors. For a given load level, two displacement configurations are possible. If the structure is loaded with a monotonically increasing force, the displacement path may jump from one configuration to another, presenting the snap-through behavior. The dynamic behavior is even richer when material non-linearities are considered. In particular, the present contribution deals with two-bar trusses made from shape memory materials.

Pseudoelasticity and shape memory are both associated with thermoelastic martensitic transformations. The shape memory effect, present in various metallic alloys, is a phenomenon where plastically deformed objects may recover their original form after going through a proper heat treatment. The pseudoelastic behavior is characterized by complete strain recovery accompanied by large hysteresis in a loading–unloading

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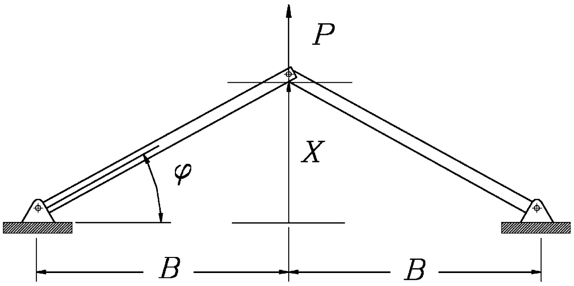


Fig. 1. Two-bars truss (von Mises truss).

cycle [2]. Fibers of shape-memory alloys (SMAs) can be used to fabricate hybrid composites exhibiting these two different but related material behaviors. Detailed description of the shape memory effect and other phenomena associated with martensitic phase transformations, as well as examples of applications in the context of smart structures, may be found in Refs. [3–8].

Adaptive trusses, with shape memory actuators for low-frequency vibration control, are examples of dynamical systems that may behave as the structure considered in this paper. The term adaptive structure has been used to identify structural systems that are capable of changing their geometry or physical properties with the purpose of performing a specific task. An adaptive structure must be equipped with actuators that induce such controlled alterations. Among the many possible choices for actuators, those made of SMAs have shown a great potential in situations where high-force, large-strain, and low-frequency structural control are needed. SMA actuators are easy to manufacture, relatively lightweight, and capable of producing high forces or displacements.

This article is concerned with the dynamic response of a shape memory two-bar truss, which is an interesting example of a structural system that exhibit both kinematic and constitutive non-linearities. A polynomial constitutive model is assumed to describe the behavior of the shape memory bars. Despite the deceiving simplicity, the authors agree that the model allows an appropriate qualitative description of the dynamical response of the system. Free and forced responses are investigated. It is shown that for a certain range of temperatures, the system may present up to 11 equilibrium configurations. Due to this rich structure, the system can easily reach a chaotic response even at moderate loads and frequencies.

## 2. Formulation

In the present investigation, we consider a shape memory two-bar truss where each bar presents the shape memory and pseudoelastic effects. The two identical bars have length $L$ and cross-section area $A$. They form an angle $\varphi$ with a horizontal line and are free to rotate around their supports and at the joint, but only on the plane formed by the two bars (Fig. 1). The critical Euler load of both bars is assumed to be sufficiently large so that buckling will not occur in the simulations reported here.

We further assume that the structure's mass is entirely concentrated at the junction between the two bars. Hence, the structure is divided into segments without mass, connected by nodes with lumped mass that is determined by static considerations. We consider only symmetric motions of the system, which implies that the concentrated mass, $m$, can only move vertically. The symmetric, vertical displacement is denoted by $X$. Under these assumptions, the balance of momentum is expressed through the following equation of motion:

$$-2F\sin\varphi + P = m\ddot{X}, \tag{1}$$

where $F$ is the force on each bar while $P$ is an external force.

There are many different works dedicated to the constitutive description of the thermomechanical behavior of shape-memory alloys, however, this is not a well-established topic. In this article, we employ a polynomial constitutive model to describe shape memory and pseudoelastic behaviors of the bars [9,10]. Despite the simplicity of this model, the authors agree that it allows an appropriate qualitative description of the dynamical response of the system. Polynomial model is concerned with one-dimensional media and proposes a sixth-degree polynomial free-energy function in terms of the uniaxial strain, $\varepsilon$. The form of the free energy is chosen in such a way that its minima and maxima are, respectively, associated with the stability and instability of each phase of the SMA. As it is usual in one-dimensional models proposed for SMAs [11], three phases are considered: austenite (A) and two variants of martensite (M+, M−). Hence, the free energy is chosen such that for high temperatures it has only one minimum at vanishing strain, representing the equilibrium of the austenitic phase. At low temperatures, martensite is stable, and the free energy must have two minima at non-vanishing strains. At intermediate temperatures, the free energy must have equilibrium points corresponding to both phases. Under these restrictions, the uniaxial stress, $\sigma$, is a fifth-degree polynomial of the strain [11], i.e.

$$\sigma = a_1(T - T_M)\varepsilon - a_2\varepsilon^3 + a_3\varepsilon^5, \tag{2}$$

where $a_1$, $a_2$ and $a_3$ are material constants, and $T$ the temperature, while $T_M$ is the temperature below which the martensitic phase is stable. If $T_A$ is defined as the temperature above which austenite is stable, and the free energy has only one minimum at zero strain, it is possible to write the following condition:

$$T_A = T_M + \frac{1}{4}\frac{a_2^2}{a_1 a_3}. \tag{3}$$

Therefore, the constant $a_3$ may be expressed in terms of other constants of the material. If the following strain definition is considered:

$$\varepsilon = \frac{L}{L_0} - 1 = \frac{\cos\varphi_0}{\cos\varphi} - 1, \tag{4}$$

the equation of motion may be rewritten as follows:

$$\begin{aligned} m\ddot{X} + \frac{2A}{L_0}X\Bigg\{ &[a_1(T - T_M) - 3a_2 + 5a_3] \\ &+ [-a_1(T - T_M) + a_2 - a_3]L_0(X^2 + B^2)^{-1/2} \\ &+ [3a_2 - 10a_3]\frac{1}{L_0}(X^2 + B^2)^{1/2} \\ &+ [-a_2 + 10a_3]\frac{1}{L_0^2}(X^2 + B^2) \\ &- \frac{5a_3}{L_0^3}(X^2 + B^2)^{3/2} \\ &+ \frac{a_3}{L_0^4}(X^2 + B^2)^2\Bigg\} = P(t), \end{aligned} \tag{5}$$

where $B$ is the horizontal projection of each truss bar (Fig. 1).

Considering a periodic excitation $P = P_0\sin(\omega t)$, and introducing a linear viscous dissipation, the equation of motion may be written in non-dimensional form as

$$\begin{aligned} x' &= y, \\ y' &= \gamma\sin(\Omega\tau) - \xi y + x\{-[(\theta - 1) - 3\alpha_2 + 5\alpha_3] \\ &\quad + [(\theta - 1) - \alpha_2 + \alpha_3](x^2 + b^2)^{-1/2} \\ &\quad - [3\alpha_2 - 10\alpha_3](x^2 + b^2)^{1/2} \\ &\quad + [\alpha_2 - 10\alpha_3](x^2 + b^2) \\ &\quad + 5\alpha_3(x^2 + b^2)^{3/2} - \alpha_3(x^2 + b^2)^2\}, \end{aligned} \tag{6}$$

where

$$x = \frac{X}{L}, \quad \omega_0^2 = \frac{2Aa_1T_M}{mL_0}, \quad \gamma = \frac{P_0}{mL_0\omega_0^2},$$

$$\xi = \frac{c}{m\omega_0}, \quad b = \frac{B}{L_0}, \quad \tau = \omega_0 t, \quad \Omega = \frac{\omega}{\omega_0},$$

$$\alpha_2 = \frac{a_2}{a_1T_M}, \quad \alpha_3 = \frac{a_3}{a_1T_M}, \quad \theta = \frac{T}{T_M},$$

$$\theta_A = \frac{T_A}{T_M} \quad \text{and} \quad ()' = \frac{\mathrm{d}()}{\mathrm{d}\tau}.$$

Numerical simulations are performed employing a fourth-order Runge–Kutta scheme with time steps chosen to be less than $\Delta\tau = 2\pi/200\Omega$. Non-linear analysis involves the determination of quantities, known as dynamical invariants, which are important to identify chaotic behavior. Lyapunov exponents have been used as the most useful dynamical diagnostic tool for chaotic system analysis. These exponents evaluate the sensitive dependence to initial conditions estimating the exponential divergence of nearby orbits. The signs of the Lyapunov exponents provide a qualitative picture of the system's dynamics and any system containing at least one positive exponent present chaotic behavior. Lyapunov exponents can also be used for the calculation of other invariant quantities as the attractor dimension, which may be determined by the Kaplan–Yorke conjecture [12]. The determination of Lyapunov exponents of dynamical system with an explicitly mathematical model, which can be linearized, is well established from the algorithm proposed by Wolf et al. [13]. Here, we employ this algorithm to calculate the Lyapunov spectrum, while the Lyapunov dimension is the parameter applied to define the attractor's dimension from the Lyapunov spectrum [13–15].

In all simulations, we have used the material properties presented in Table 1. These values were chosen in order to, as shown in Fig. 2, match the experimental data obtained by Sittner et al. [16] for a Cu–Zn–Al–Ni alloy at 373 K. For the data in Table 1, the parameters

Table 1
Material properties

| $a_1$ (MPa/K) | $a_2$ (MPa) | $a_3$ (MPa) | $T_M$ (K) | $T_A$ (K) |
|---|---|---|---|---|
| 523.29 | $1.868 \times 10^7$ | $2.186 \times 10^9$ | 288 | 364.3 |

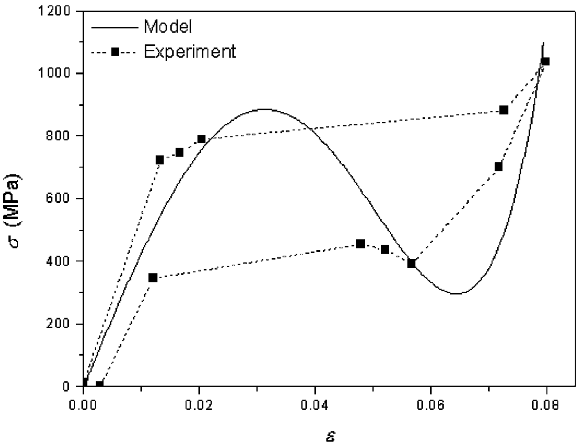


Fig. 2. Stress–strain curve: experimental and predicted by polynomial model.

defined in Eq. (6) assume the values: $\alpha_2 = 1.240 \times 10^2$ and $\alpha_3 = 1.450 \times 10^4$. We further let $b = 0.866$, corresponding to a two-bar truss with an initial position $\varphi_0 = 30^\circ$.

## 3. Free vibration

In this section, we discuss the free response of the shape memory two-bar truss. This is done by letting $\gamma$ vanish in the equations of motion (6). It is well known that the von Mises truss presents three equilibrium points due to kinematics non-linearity [1]. Of those, two are stable while the other one is unstable. In the case of a shape memory two-bar truss, constitutive non-linearity introduces a different behavior. Denoting by $(\bar{x}, \bar{y})$ a point that makes the right-hand sides of equations of motion vanish, the following possibilities are found:

$$\bar{x} = 0 \quad \text{and} \quad \bar{y} = 0, \tag{7}$$

$$\bar{x} = \pm\sqrt{1 - b^2} \quad \text{and} \quad \bar{y} = 0,$$

$$\bar{x} = \pm\sqrt{\frac{1}{2\alpha_3}[\sqrt{\chi_1} \pm \sqrt{8\alpha_3(\alpha_2 + \sqrt{\chi_1}) + \chi_2}]}$$

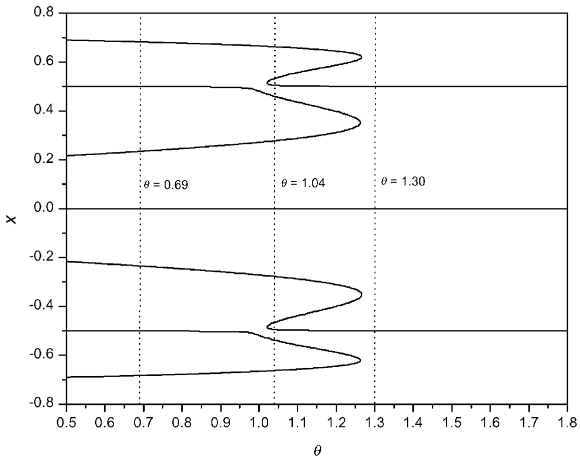


Fig. 3. Map of equilibrium points as a function of temperature.

$$\text{and} \quad \bar{y} = 0,$$

$$\bar{x} = \pm\sqrt{\frac{1}{2\alpha_3}[-\sqrt{\chi_1} \pm \sqrt{8\alpha_3(\alpha_2 - \sqrt{\chi_1}) + \chi_2}]}$$

$$\text{and} \quad \bar{y} = 0,$$

where $\chi_1 = \alpha_2^2 - 4\alpha_3(\theta - 1)$ and $\chi_2 = 2\alpha_3(1 - b^2) + \alpha_2$. Of these 11 possibilities, only those that correspond to real numbers have physical meaning. Stability of these equilibrium configurations may be determined by the behavior of the system in their neighborhood. An analysis of the eigenvalues of the Jacobian matrix of the system reveals its local stability. Therefore,

(a) $0 < \theta < 1$ ($\alpha_2^2 < \chi_1 < 4\alpha_3 + \alpha_2^2$): the system has seven fixed points. The origin of the phase space and $\bar{x} = \pm\sqrt{1 - b^2}$ are saddle points. The other four fixed points are centers. This is consistent with the low-temperature behavior of SMA, where two martensitic phases are stable.

(b) $1 \leqslant \theta < \theta_A$ ($0 < \chi_1 \leqslant \alpha_2^2$): the system has 11 fixed points, the five being saddle points and the remaining are centers. The existence of six stable fixed points is explained by the stability of both martensitic phases and austenite in this range of temperature.

(c) $\theta \geqslant \theta_A$ ($\chi_1 \leqslant 0$): the system has three fixed points. The origin and $\bar{x} = \pm\sqrt{1 - b^2}$ are centers. Under this temperature range, austenite is the only stable phase in the stress-free SMA.

A map of equilibrium points as a function of temperature is presented in Fig. 3 for Cu–Zn–Al–Ni alloy. At low temperatures, where the martensitic phase is stable

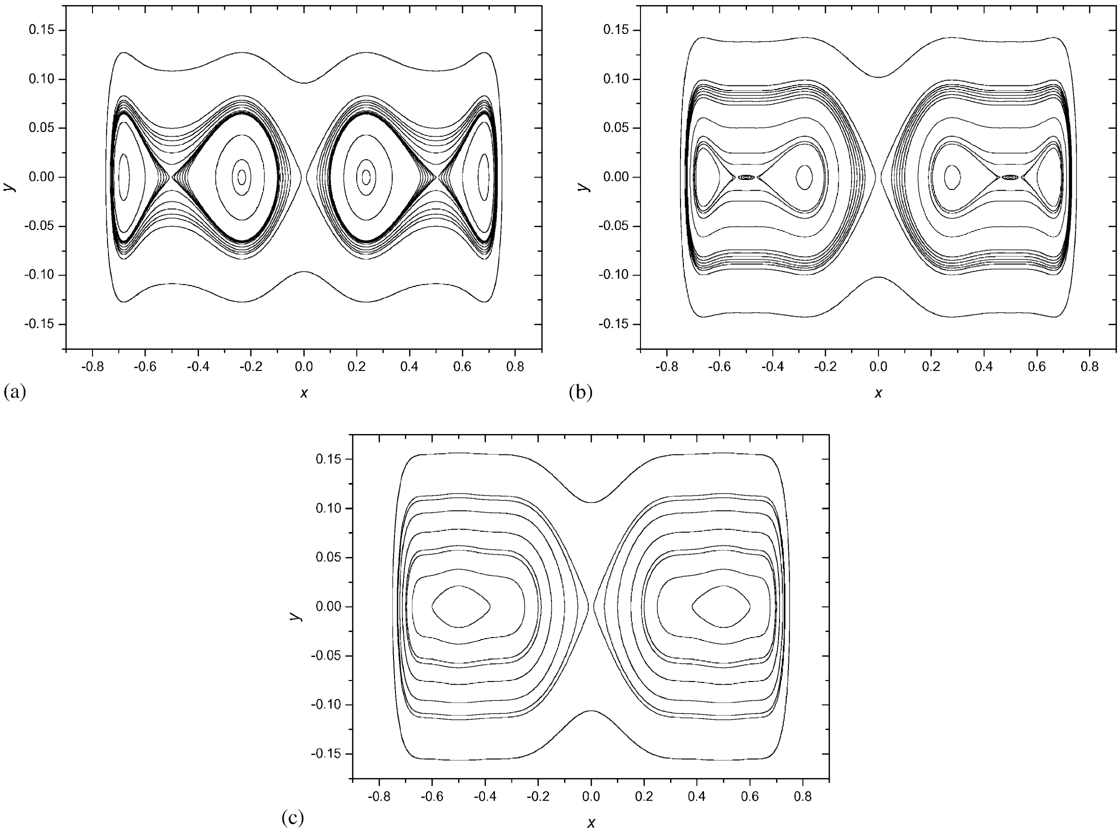


Fig. 4. Phase portrait (a) $\theta = 0.69$; (b) $\theta = 1.04$; (c) $\theta = 1.30$.

$(0 < \theta < 1)$, there are seven equilibrium points, four of them stable while the others are unstable. By considering a higher temperature, where both martensitic and austenitic phases may coexist $(1 \leqslant \theta < \theta_A)$, the system exhibits five unstable and six stable equilibrium points. At an even higher temperature, where only the austenitic phase is present $(\theta \geqslant \theta_A)$, the system has one unstable and two stable equilibrium points.

In order to illustrate the free response of the shape memory truss, we consider the non-dissipative system, that is, we let $\xi = 0$ in Eq. (6). Results from simulations are presented in the form of phase portraits. Fig. 4 presents the free response of the system at different temperatures. Fig. 4a considers a temperature where the martensitic phase is stable $(\theta = 0.69)$. There are, in this case, seven equilibrium points. From these, four are stable while the other three are unstable. At a higher temperature, when martensitic and austenitic phases are both present in the alloy $(\theta = 1.04)$, the system, as depicted in Fig. 4b, has five unstable and six stable equilibrium points. Fig. 4c is representative of the free response at higher temperatures $(\theta = 1.30)$, when the alloy is fully austenitic. As discussed above, in this case, the system has two centers and a single saddle point.

## 4. Forced vibration

In this section, we address the forced vibration response. The dynamic behavior is now much richer. Periodic, quasi-periodic and chaotic motions are all possible in the two-bar truss. In order to start the analysis, let us consider the bifurcation diagram which represents stroboscopically sampled displacement values, $x$, under the slow quasi-static increase

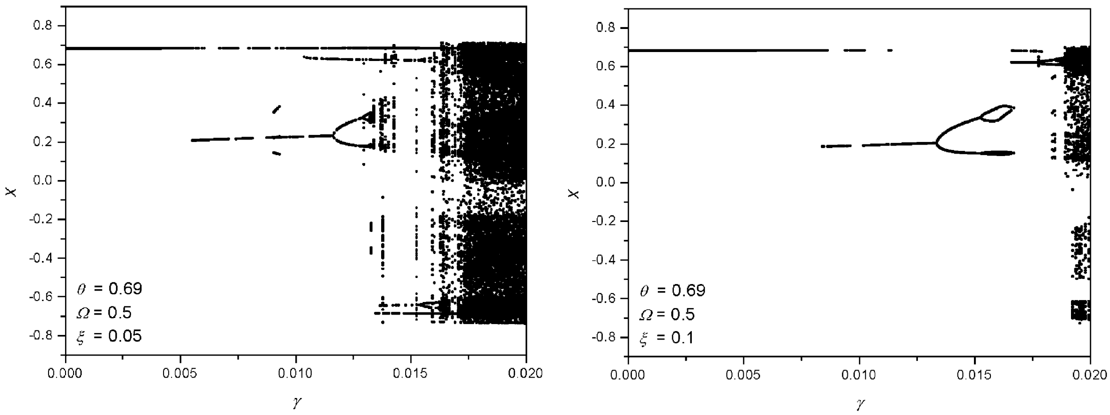


Fig. 5. Bifurucation diagrams varying $\gamma$ for $\theta - 0.69$ and $\Omega = 0.5$.

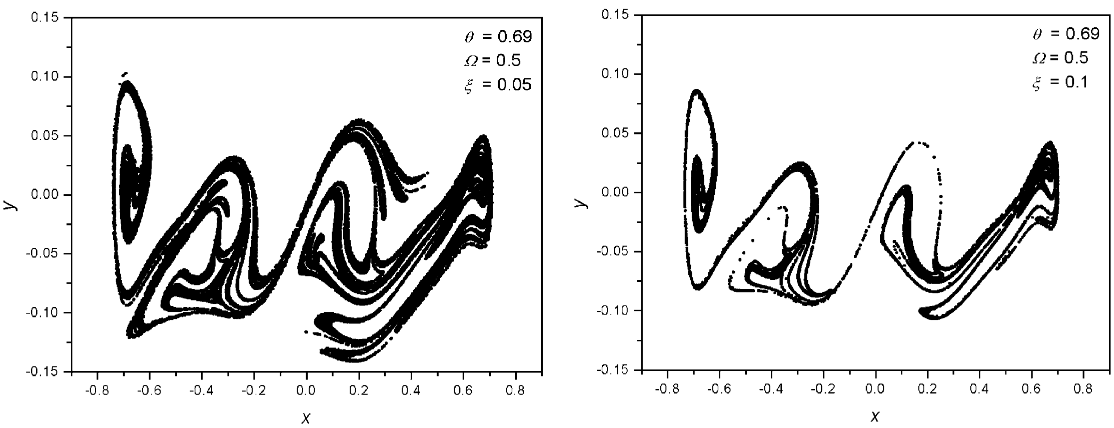


Fig. 6. Strange attractors ($\theta = 0.69$, $\Omega = 0.5$, $\gamma = 0.020$).

of the driving force amplitude, $\gamma$. Different temperature, frequency and dissipation parameters are considered.

At first, we consider a temperature $\theta = 0.69$ where the martensitic phase is stable. The frequency parameter, $\Omega$, is fixed at 0.5, while different values for dissipation are adopted. Fig. 5 shows bifurcations and clouds of points related to chaotic motions.

When $\Omega$=0.5 and $\gamma$=0.020, the two-bar truss exhibit chaotic motion. Poincaré mapping eliminates one dimension of the system by sampling the displacement and velocity stroboscopically at time intervals, summarizing the dynamical system behavior. Fig. 6 shows Poincaré sections of these motions for two different dissipation parameters ($\xi = 0.05$ and 0.1). Both situations present strange attractors with fractal dimension, which may be evaluated from the Kaplan–Yorke conjecture [12,13]. The Lyapunov spectrum associated with $\xi = 0.05$ is $\lambda_i = (+0.14, -0.21)$ and the Lyapunov dimension is $D = 1.66$. When $\xi = 0.1$, we obtain $\lambda_i = (+0.09, -0.23)$ and $D = 1.38$. As expected, the system's attractor with higher dissipation tends to occupy a small region in phase space and the attractor dimension is smaller.

Now, a higher temperature is considered and the austenitic phase is stable ($\theta = 1.30$). Bifurcation diagrams presented in Fig. 7 shows the influence of the variation of the parameter $\gamma$ for different dissipation values. Note that for higher values of $\xi$, the behavior of the two-bar truss tends to be more regular.

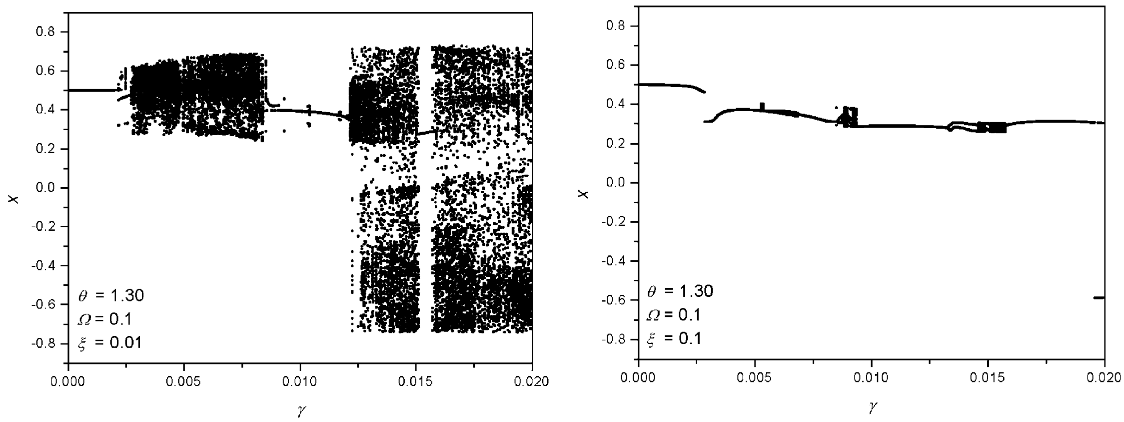


Fig. 7. Bifurcation diagrams varying $\gamma$ for $\theta = 1.30$ and $\Omega = 0.1$.

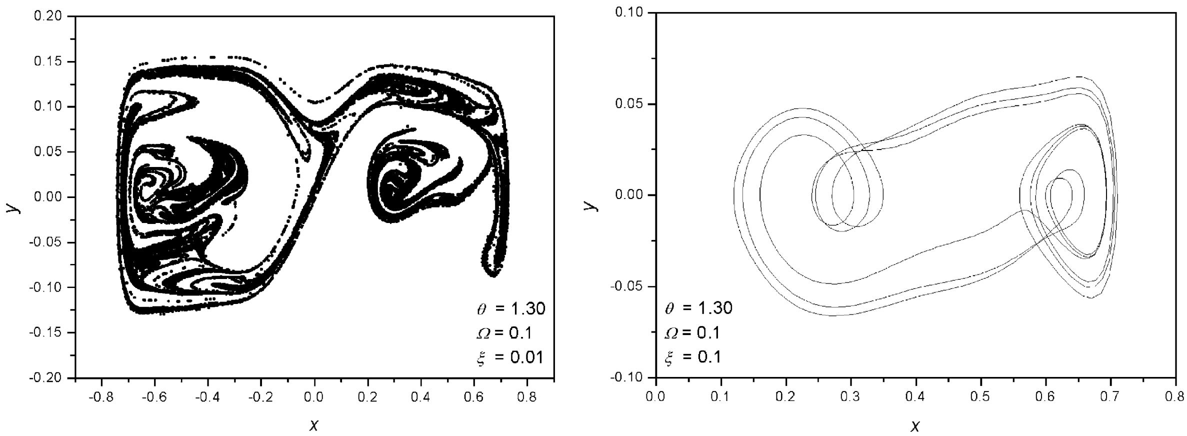


Fig. 8. Response for $\theta = 1.30$, $\Omega = 0.1$, $\gamma = 0.015$. Strange attractor ($\xi = 0.01$) and period-3 motion ($\xi = 0.1$).

Let us consider an excitation with $\Omega = 0.1$ and $\gamma = 0.015$. With these parameters the two-bar truss presents different behaviors for $\xi = 0.01$ and 0.1 (Fig. 8). When $\xi = 0.01$, the system behaves chaotically and the Lyapunov spectrum is characterized by $\lambda_i = (+0.02, -0.04)$, while $D = 1.59$. When $\xi = 0.1$, a period-3 motion occurs.

Considering a temperature where both martensite and austenite coexist, the strange attractor changes. Fig. 9 shows the strange attractor related to the motion where $\theta$=1.04, $\Omega$=0.1, $\gamma$=0.015 and $\xi$=0.01. The Lyapunov spectrum of this motion is $\lambda_i = (+0.03, -0.05)$ while $D = 1.68$. As expected, this attractor has a dimension greater than that at higher temperature. This may be attributed to the elimination of equilibrium points when the temperature increases. Increasing the dissipation ($\xi = 0.1$) the system presents a period-3 motion.

In order to obtain a better comprehension of the effect of temperature on the behavior of the two-bar truss, we consider bifurcation diagrams with respect to the parameter $\theta$. Fig. 10 illustrates two different situations. At the left diagram, we consider $\gamma = 0.015$, $\Omega = 0.1$, and $\xi = 0.01$. In the other plot, the chosen values were $\gamma = 0.020$, $\Omega = 0.5$, and $\xi = 0.05$. Note that there is an inversion in the range of chaotic motion.

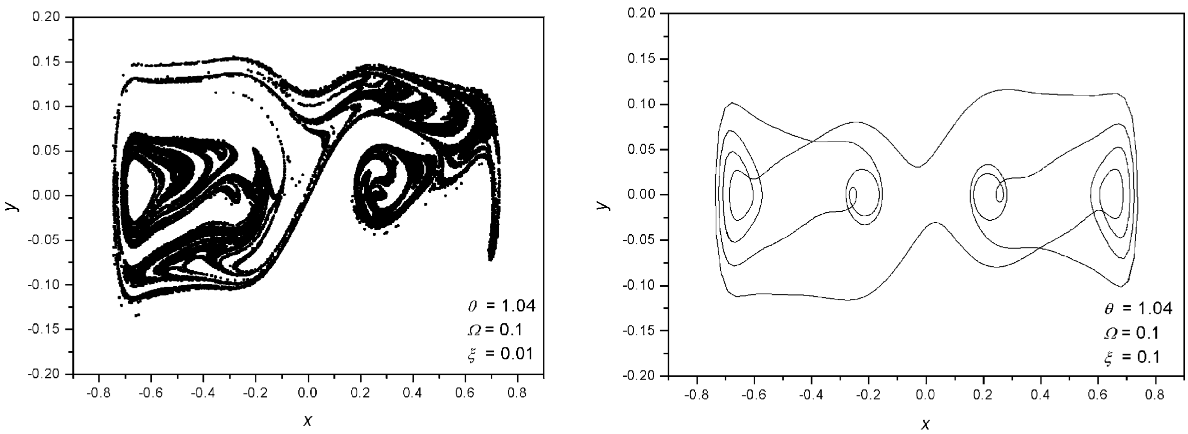


Fig. 9. Response for $\theta = 1.04$, $\Omega = 0.1$, $\gamma = 0.015$. Strange attractor ($\xi = 0.01$) and period-3 motion ($\xi = 0.1$).

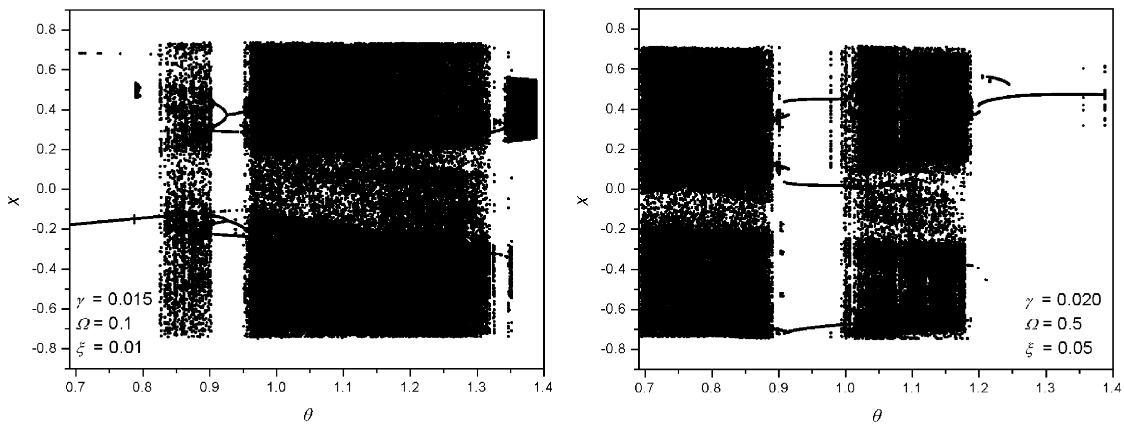


Fig. 10. Bifurcation diagrams varying $\theta$.

In the first case, chaos occurs at higher temperatures, and the kinematics non-linearity is dominant. In the other case simulated here, chaos occurs at low temperatures, meaning that the constitutive non-linearity is preponderant.

## 5. Conclusions

This article reports results from numerical simulations of the dynamical response of a shape memory two-bar truss. A polynomial constitutive model was assumed to describe the constitutive behavior of the bars. Despite the deceiving simplicity, the authors agree that the model allows an appropriate qualitative description of the dynamical response of the system. Free and forced responses were investigated. Numerical simulations were carried on employing a fourth-order Runge–Kutta scheme for numerical integration, and the characterization of chaos was performed with Lyapunov exponents, evaluated with the aid of an algorithm proposed by Wolf et al. (1985). Results have shown that the system may present a number of interesting, complex behaviors. Basically, the system response is a combination of kinematics and constitutive non-linearities. Due to this combination, it may present 11 equilibrium points, which are responsible for a very rich dynamics. At higher temperatures, the kinematics non-linearity is dominant while at low temperatures, the constitutive non-linearity is preponderant. Strange attractors of this system vary their pattern with temperature.

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