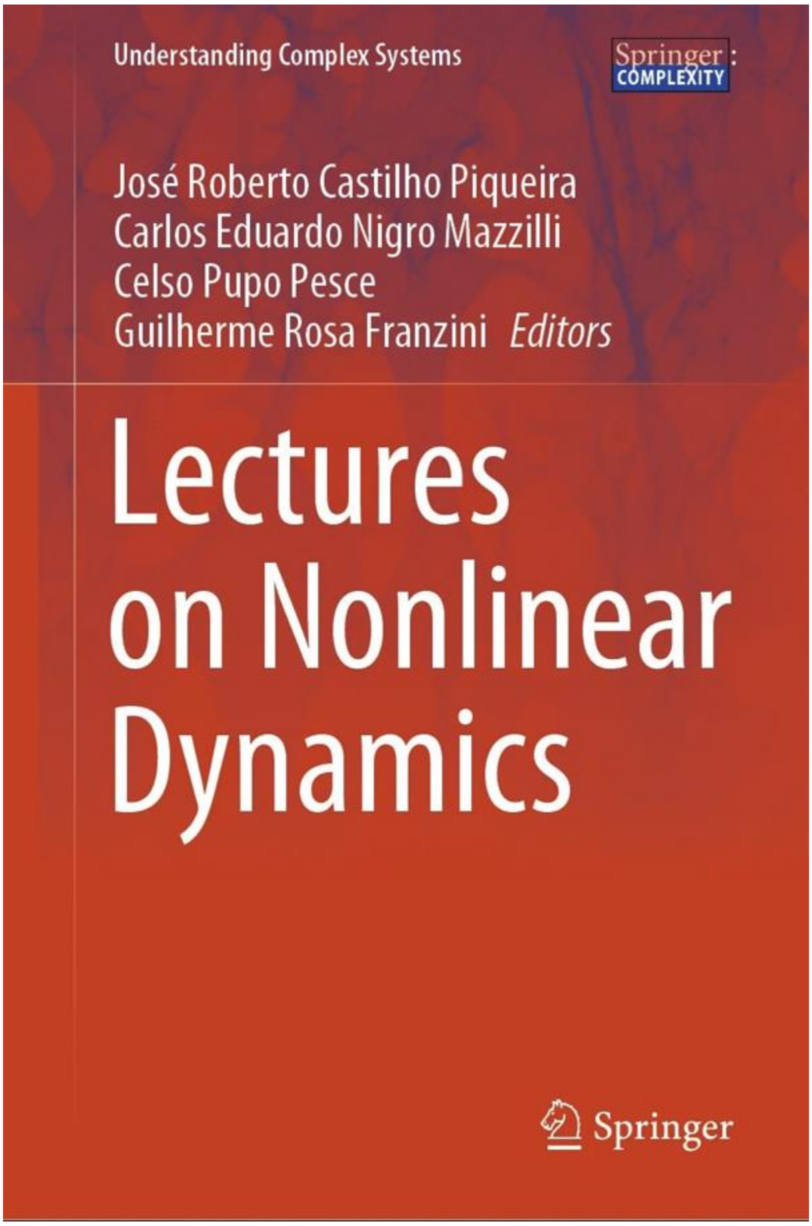

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Chaos Theory

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Abstract This chapter presents an overview of chaos theory. It starts from the background of dynamical systems, presenting the mathematical representation and the concept of stability. Afterward, chaotic dynamics is explored from the horseshoe transformation, establishing that it is a consequence of the contraction-expansion-fold process. The main aspects of chaotic behavior are then discussed defining chaotic and fractal attractors. Routes to chaos are investigated showing some definitions of bifurcation, treating local and global bifurcations. Lyapunov exponents are defined in order to present a diagnostic tool for chaos.

Keywords Chaos; nonlinear dynamics; dynamical systems; stability; bifurcation; horseshoe transformation; Lyapunov exponents.

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1 Introduction

Nonlinearities are the essential characteristics responsible for a great variety of possibilities of natural systems. Rhythms are dynamical manifestations of the natural system behavior with an intrinsic richness expressed by regular and irregular responses over time and space. These ideas motivate dynamical investigations in different areas of human knowledge, varying from mechanics to biology.

The scientific revolution is symbolic represented by Galileo Galilei (1564-1642) who introduced the idea of the experimental verification as the main source of the truth and the mathematics as the alphabet of the universe. Isaac Newton (1643-1727) consolidated these revolutionary ideas establishing the laws of motion and the mathematical description of the phenomena from *governing equations*.

During this scientific revolution, nonlinear systems were usually avoided creating a linear paradigm that limited the human comprehension about natural processes. One of these paradigms is the strict determinism, clearly illustrated by the Pierre-Simon Laplace (1749-1827) thinking: "*If we conceive of an intelligence which at a given instant comprehends of all the relations of the entities of this universe, it could state the respect positions, motions, and general effects of all these entities at any time in the past or future*". Philosofically, the strict determinism questioned the free will, establishing that nothing can be changed after the definition of the "*relations of the entities of the universe*" or, in other words, the *governing equations*.

The strict determinism and the linear paradigm started to be broken only in the end of the XIX century. Motivated by the stability analysis of the universe, Jules Henri Poincaré (1854-1912) studied the dynamical response of the three-body problem. Poincaré presented a counterpoint for the strict determinism of Laplace: "*Even if the case that the natural laws had no longer secret for us, ... it may happen that small differences in initial conditions produce very great ones in the final phenomena*". This is the essential characteristic of nonlinearity that means that small causes may generate great effects.

Chaos is a possible kind of response of a nonlinear system being characterized by sensitive dependence on initial conditions. Hence, a nolinear system has a rich dynamic including regular and irregular behaviors. Although Poincaré has an absolutely clear vision with respect to chaos (as it is understood nowadays), only in 1963, when Edward Lorenz (1917-2008) developed meteorology studies, this idea came back to the scientific scenario (Lorenz, 1963). The Lorenz's analysis associated the sensitive dependence of initial conditions on the idea of the *butterfly effect*, which means that if a butterfly flaps its wings in China it may cause a hurricane in Brazil.

Afterward, nonlinear dynamics and chaos concepts started to be incorporated for the proper investigation of several subjects, being applied in all areas of human knowledge passing through engineering, mechanics, chemistry, biology, economy, psychology, among others. This new approach represented the dynamical freedom, bringing for the analysis ideas hidden by the linear thinking (Stewart, 1991; Mullin, 1993; Lorenz, 1995; Gleick, 1997; Savi, 2014).

This chapter presents a general overview of the chaos theory, providing a formal comprehension of the chaotic dynamics. Initially, dynamical systems background is provided, presenting their mathematical representantion and the concept of stability. The formal definition of chaos is treated in the sequence using the idea of the horseshoe transformation. Chaotic and fractal attractors are discussed. Lyapunov exponents are then presented as a diagnostic tool of chaos.

2 Dynamical systems: background

A dynamical system is a frame-by-frame description of reality being represented by a transformation f imposed to state variables x – employed to describe a phenomenon, defining a vector field. Its mathematical description is a set of differential equations as follows:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \ \ \boldsymbol{x} \in \boldsymbol{R}^{\boldsymbol{n}} \tag{1}$$

This system is called *autonomous* since it does not have an explicit dependence on time. On the other hand, it is possible to consider a *non-autonomous* system that has an explicit time dependence as follows:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{t}), \ \ \boldsymbol{x} \in \boldsymbol{R}^{\boldsymbol{n}} \tag{2}$$

A non-autonomous system can be represented by an autonomous system by increasing the system dimension.

Dynamical system analysis can be performed by a geometrical perspective, usually called *topology*, studying continuous transformations. In this regard, consider that the space of dependent variables, x, called *state space* or *phase space*, has different topologies, characterizing several dynamical aspects. An interesting approach to understand the system dynamics is to monitor an object in phase space, observing how it evolves under transformations imposed by *f*. This object can be built by considering different initial conditions and, for each one of these conditions, there is an *orbit* or *trajectory*, evolving from one frame to the subsequent ones. These trajectories are the solution of the differential equations and, based on this, a new object is formed in a subsequent specific frame. Fig. 1 presents the frame-by-frame description showing an object on one frame evolving for a new object in the new frame.

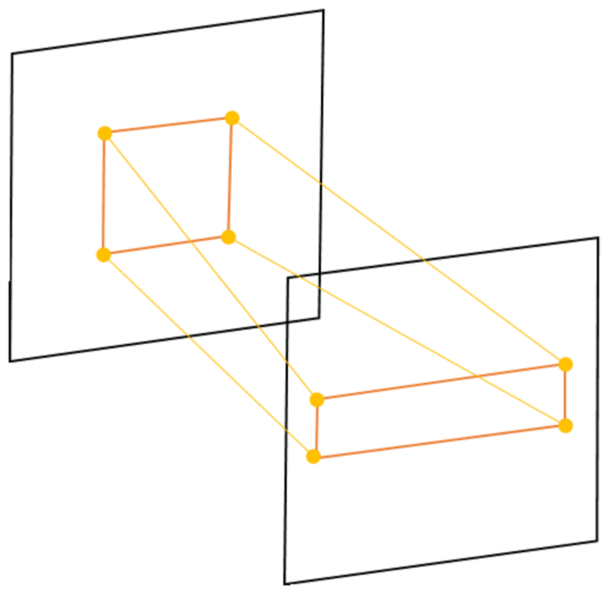


Fig. 1 Dynamical system $\dot{x} = f(x),\ x \in R^n$ represented by a frame-by-frame description showing how an object evolve to a new one.

Since nonlinear systems usually do not have analytical solution, they need to be treated with proper tools. Numerical procedures are usually employed for this aim. The fourth-order Runge-Kutta method is an example of a numerical procedure to obtain the desired solution (Savi, 2017). Perturbation techniques are also an interesting alternative to solve nonlinear systems (Nayfeh & Mook, 1979). Special procedures should also be employed to treat unstable solutions since the so-called brute force integration is capable to capture just the stable solutions.

2.1 *Stability*

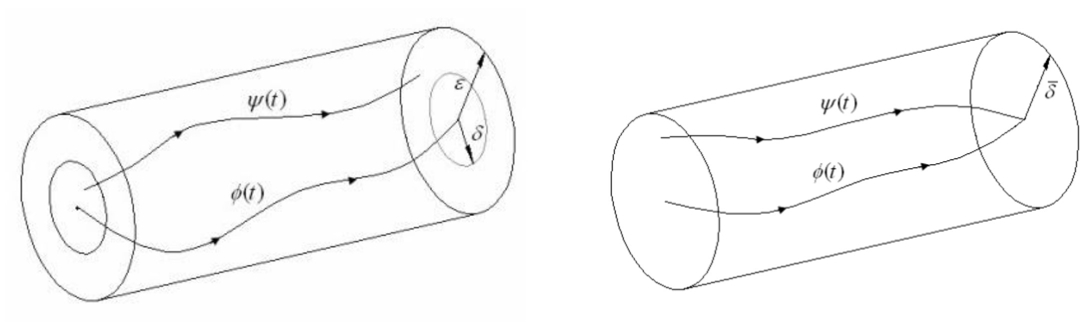
Stability is an essential issue of nonlinear dynamical systems being associated with characteristics of a solution subjected to perturbations. If a perturbation does not affect a system response in a significant way, the system is *stable*. Otherwise, the system is *unstable*.

Alexander Lyapunov (1857-1918) developed a stability theory for dynamical systems establishing a relationship between a specific orbit or solution and its perturbation, represented by a nearby orbit associated with different initial conditions in the neighborhood of the original one. The stablity concept of Lyapunov defines a stable system in such a way that two nearby orbits remain close to each other with the evolution of time. Fig. 2 shows the idea of the Lyapunov stability showing that the system is stable if there is a $\delta = \delta(\varepsilon) > 0$ in such a way that:

$$\text{If } |\boldsymbol{\psi}(\boldsymbol{t_0}) - \boldsymbol{\phi}(\boldsymbol{t_0})| < \boldsymbol{\delta} \quad \text{then } |\boldsymbol{\psi}(\boldsymbol{t}) - \boldsymbol{\phi}(\boldsymbol{t})| < \boldsymbol{\varepsilon} \qquad (3)$$

The system is called asyntoptically stable if these orbits tend to converge to each other when time tends to infinity, being defined by $\bar{\delta} > 0$ in such a way that (Fig. 2b):

$$\text{If } |\psi(t_0) - \phi(t_0)| < \bar{\delta} \quad \text{then } \lim_{t\to\infty} |\psi(t) - \phi(t)| = 0 \qquad (4)$$



(a) (b)

Fig. 2 Stability concept of Lyapunov. (a) Stable; (b) asytoptically stable.

3 Chaos

Nonlinear dynamical systems present a great variety of responses that can be understood as a system freedom, associated with rich behaviors. Chaos is one of these possibilities related to richness and unpredictability. In brief, chaos may be defined as the apparent sthocastic behavior of deterministic systems (Thompsom & Stewart, 1986; Kapitaniak, 1991; Moon, 1992; Ott, 1993; Strogatz, 1994; Alligood *et al.*, 1997; Savi, 2017).

Since a dynamical system may be understood as a transformation f that is imposed to state variables x, it is interesting to imagine a special type of transformation characterized by a sequence of contraction-expansion-fold process that represents an archetypal behavior of the system being called *horseshoe transformation*. This transmormation was originally proposed by the mathematician Steve Smale, and because of that, it is sometimes called as the *Smale horseshoe*. In order to understand the horseshoe transformation, consider an object in the phase space - a unitary square – subjected to a contraction-expansion-fold process illustrated in Fig. 3 as a frame-by-frame description.

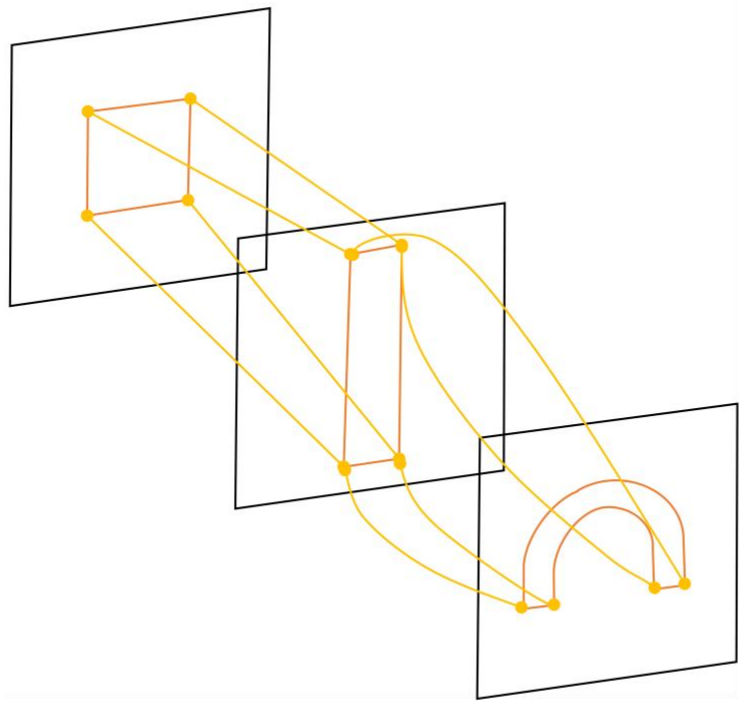


Fig. 3 Frame-by-frame description of the horseshoe transformation.

As a matter of fact, two different transformations can be imagined: a positive part of transformation, *f*, and a negative part, f^{-1}, represented in Fig. 4. The positive part has the expansion on the vertical direction while the contraction occurs on the horizontal direction. The folding process occurs in the sequence. On the other hand, the negative part is the opposite, presenting the expansion on the horizontal direction and the contraction on the vertical. The folding is similar. The limit as the number of interactions of these transformations tends to infinity, produces a set of vertical lines for the positive part of transformation and a set of horizontal lines for the negative part. It is noticeable that these lines can be identified by a sequence of 0's and 1's.

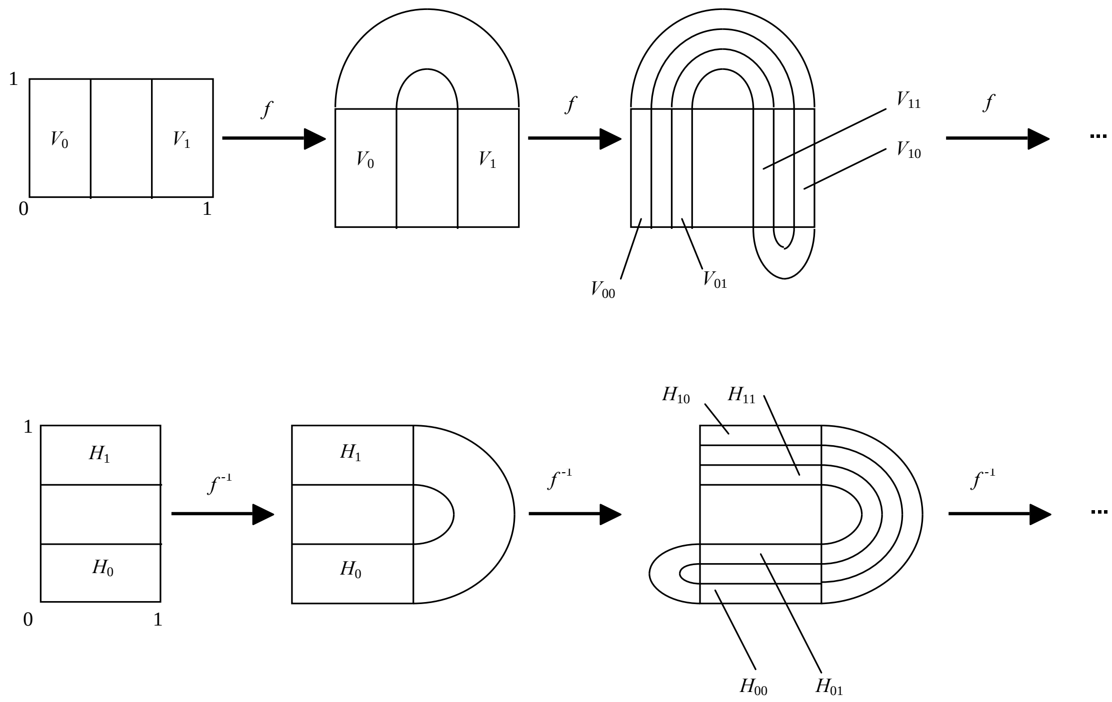


Fig. 4 Horseshoe transformation associated with contraction-expansion-fold process.

An invariant set of points can be built by the intersection of positive and negative parts of the horseshoe transformations. Since a generic point of this invariant set is an intersection of vertical and horizontal lines, it can also be identified by a sequence of 0's and 1's. Because of that, it is possible to build a structure that represents orbits of dynamical systems from these binary sequences. This approach is called *symbolic dynamics* and, since it is based on sequences of integer numbers, it is not associated with floating point errors, being useful in several situations. Symbolic space has a topological equivalence with the real one, and therefore, it is possible to analyze the system dynamics from the symbolic space.

In order to establish an interpretation of the horseshoe transformation, it should be pointed out that each one of the effects of the contraction-expansion-fold process represents three distinct directions respectively associated with stable, unstable and neutral chararacteristics. In this regard, there is an unstable direction that promotes a divergence between two close points. This unstable direction characterizes the sensitive dependence on initial conditions.

On this basis, consider two generic points on the symbolic space, p_1 and p_2, Fig. 5, that belong to a small neighborhood, ε. These points are associated with counterparts on the real space. Due to the characteristics of the horseshoe transformation, it does not matter how close these points are, or in other words, how small is the neighborhood, there is a finite number of iterations that makes these points to be separated by a finite distance. This means that two close points on phase space have sensitive dependence on initial conditions.

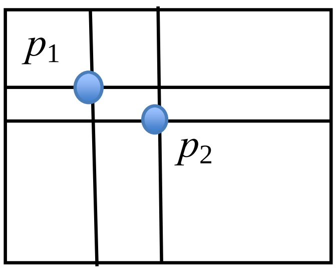


Fig. 5 Two points on the invariant set associated with the horseshoe transformation.

The existence of the horseshoe transformation is the fundamental dynamical characteristic of chaos. Therefore, chaos needs to be associated with nonlinear systems with, at least, three dimensions, related to distinct directions: contraction, expansion and a neutral direction, where fold occurs (Savi, 2017; Wiggins, 1990; Guckenheimer & Holmes, 1983). Due to the unstable direction, associated with expansion process, chaotic behavior has a sensitive dependence on initial conditions, which establishes that small causes are related to great effects, the holistic image of the butterfly effect. Fig. 6

presents a schematic picture that shows the connection of the horseshoe transformation and the sensitive dependence on initial condtions.

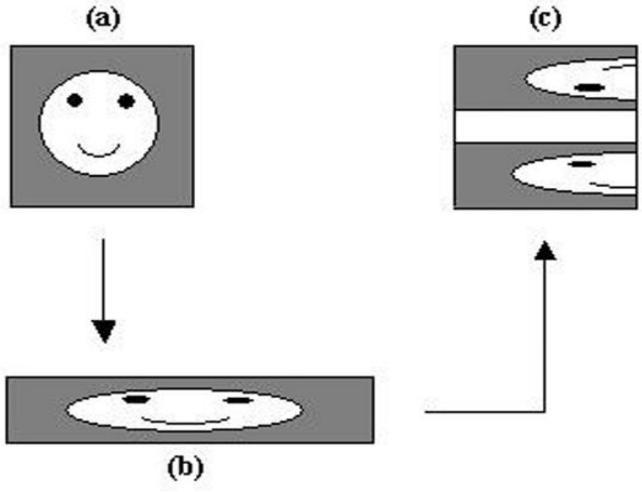


Fig. 6 Schematic representation of the sensitive dependence on initial conditions associated with horseshoe transformation

By observing a chaotic behavior in phase space, it is possible to observe open orbits that never closes. Time history of state variables have non-periodic characteristic. Fig. 7 presents system dynamics of the Lorenz system that is a 3-Dim system that describes the Rayleigh-Benard fluid convection, which contemplates two parallel plates, separated by a fluid, where the upper plate has a lower temperature when compared with the lower plate. Tinny variations on initial conditions causes dramatic changes on system responses, with a clear divergence of nearby orbits.

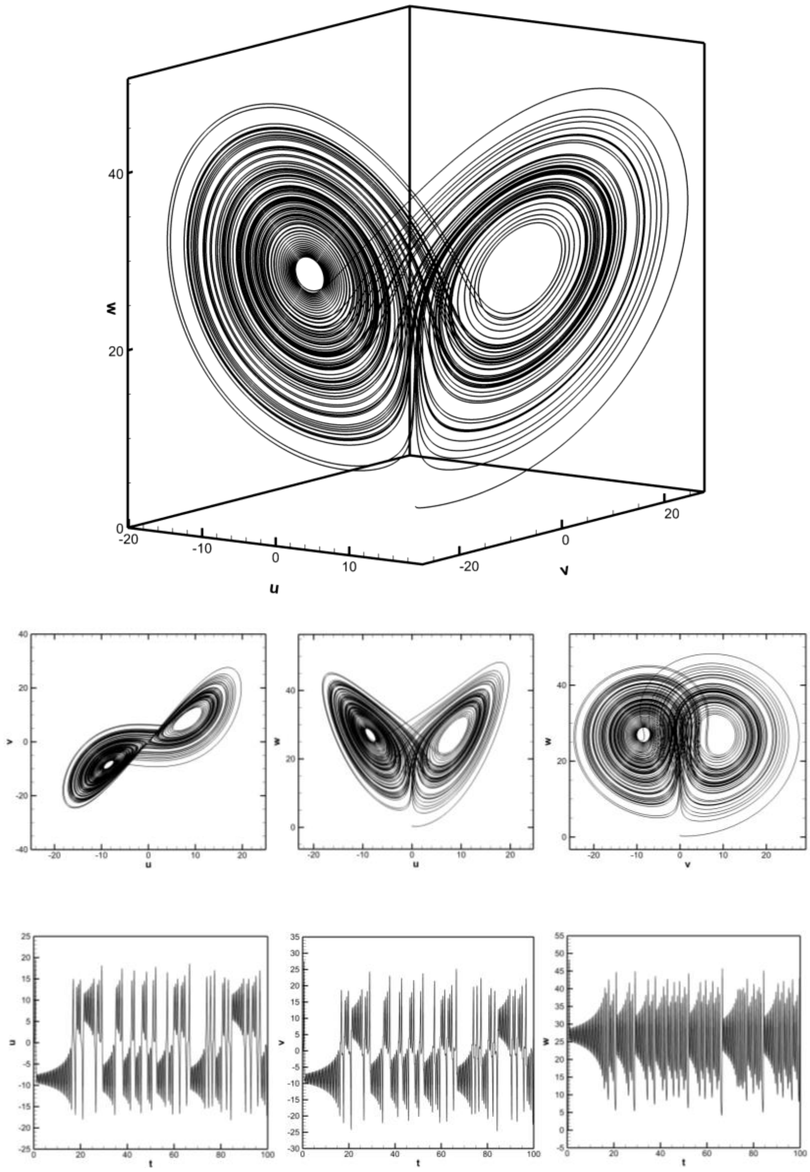


Fig. 7 Lorenz system showing a typical chaotic orbit represented by the phase space (upper panel), phase subspaces (central panel) and state variable time history (lower panel).

3.1 Chaotic attractors

The horseshoe transformation invariant set has the structure of the *Cantor set* that is closed, disconnected with an uncountable infinity of points. An essential example of this set is shown in Fig. 8 considering a line that is split into three equal parts and the center part is discarded. This process is reapeted and, when the process tends to infinity, a disconnected set of points is generated. Since the original line is a typical 1-Dim structure and the set generated by the process has disconnected characteristic - that is not a point, it is possible to infer that it has a fractional dimension, between 1 and 0. This kind of structure has a *fractal* characteristic as a reference of the non-integer, fractional nature of its dimension.

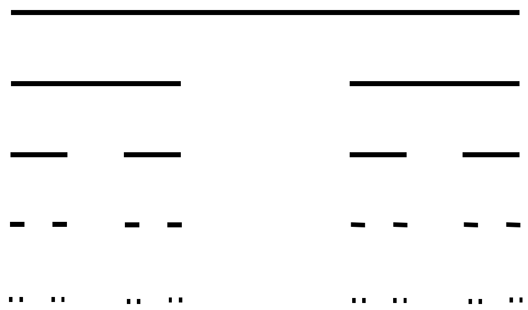


Fig. 8 Cantor set – one dimensional structure built with the rule that considers a line that is split into three equal parts and the center part is discarded.

The visualization of the chaotic behavior is interesting to be performed with the aid of Poincaré map that is a stroboscopic view of the system dynamics. There are several possibilities to build a map but, it reduces the system dimension by a transformation. A secant section in the phase space is a strategy that allows to observe the time instants when orbits transversally cross the section. Another situation is related to a periodic external excitation that defines a frequency of the stroboscopic view. Fig. 9 presents a geometric view of the idea of the Poicaré map construction.

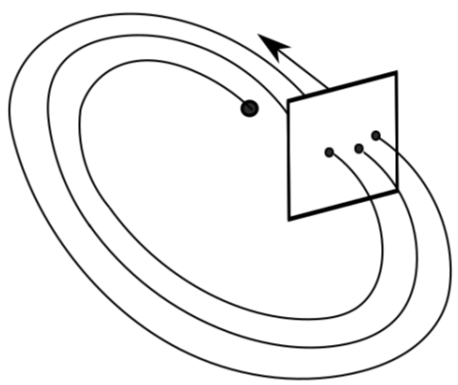


Fig. 9 Poincaré map.

In order to illustrate the physical behavior of the horseshoe transformation, consider an object represented by a circle of initial conditions. After some interactions, it is evaluated by the intersection of the orbits on a Poincaré section. If the system presentes a chaotic behavior, it is associated with a horseshoe transformation and therefore presents contraction-expansion-fold process. Fig. 10 shows the evolution of a circle of initial conditions in different time instants, or different Poicaré sections, considering a chaotic response. Periodic response is presented in Fig. 11. Since the system presents a regular, periodic behavior, there is not an expansion process, which means that orbits are convergent.

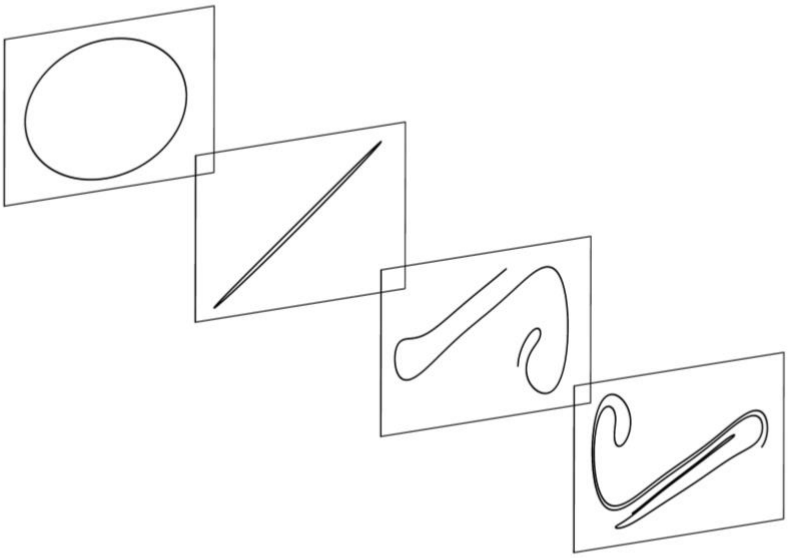


Fig. 10 Chaotic evolution of a circle of initial conditions.

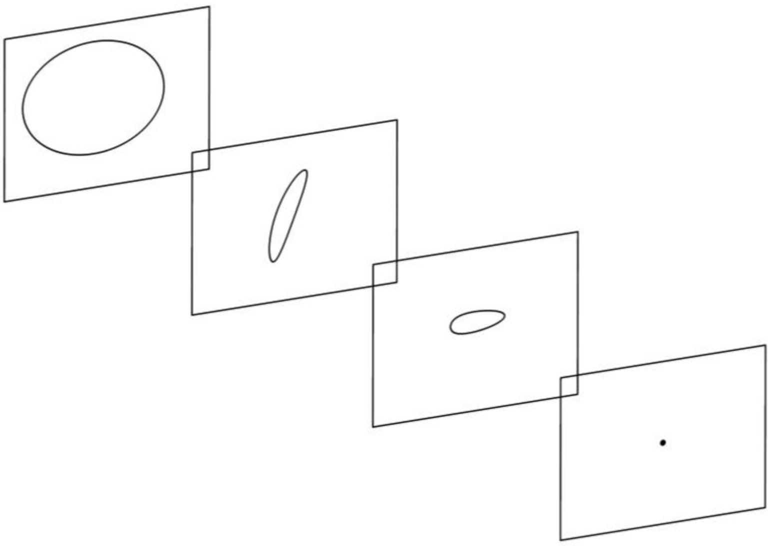


Fig. 11 Periodic evolution of a circle of initial conditions.

Dissipative dynamical systems are characterized by asymptotic behavior, being associated with attractors. Several types of attractors can be observed in dynamical system. A stable equilibrium point can be understood as a 0-Dim attractor. A limit-cycle is another possibility of a 1-Dim attractor. Chaotic behavior is also related to an attractor that represents a preferenced region of the phase space where orbits converge. A chaotic attractor represents a collection of points, organized in lamelas, with voids, being associated with a Cantor set produced by the horseshoe transformation. Due to that, the name *strange attractor* is usually employed where its strangeness is related to a geometrical aspect, essentially fractal, with non-integer dimension. Chaoticity, on the other hand, is a dynamical aspect. Therefore, although not usual, it is possible to have different situations with respect to chaos in dynamical systems: chaotic strange attractor; chaotic non-strange attractor; strange non-chaotic attractor (Grebogi *et al.*, 1984). A typical chaotic, strange attractor is presented in Fig. 12 showing different Poincaré sections. A careful observation of these attractors allows one to see the horseshoe transformation applied to the attractor set of points (Savi, 2017).

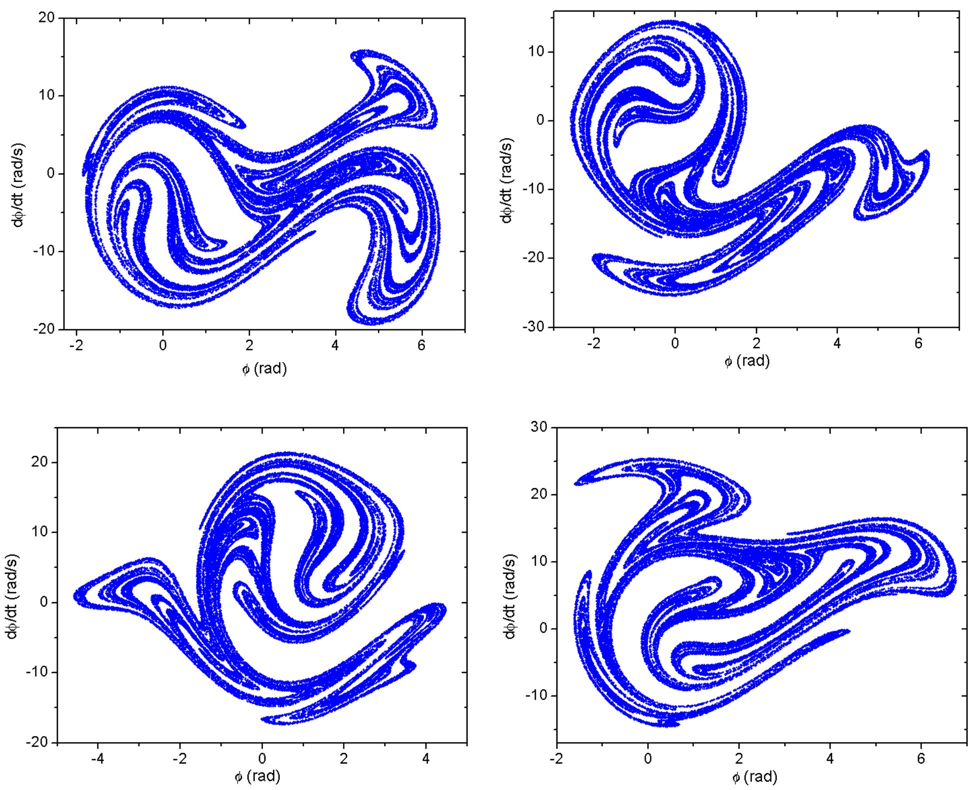


Fig. 12 Chaotic strange attractor – different Poincaré sections.

Chatotic behavior has the sensitive dependence on initial conditions as its essential characteristic. But the fundamental structure behind chaos richness is the existence of an infinite number of unstable periodic orbits embedded in the chaotic behavior. This means that the complex chaotic orbit is a consequence of the visit to an infinity number of periodic orbits and, since they are unstable, the system visits each one of them, being expelled in the sequence. This characteristic confers flexibility to the chaotic behavior, being one of its essential aspects. The common sense knows this kind of idea when analyzes human behavior. Huge changes are easier when your life is a mess, chaotic. A well-established life, with periodic characteristics, is difficult to be changed. This points to the advantages of the use chaos instead of resist to it (Briggs & Peat, 2000).

4 Routes to Chaos

The different responses of a dynamical system are defined by parameters and initial conditions. Each set of parameter produces a specific response, and a proper comprehension of system dynamics includes the form of how system behavior is altered by parameter changes. Poincaré introduced the idea of qualitative changes in solution structure using the term *bifurcation*. Multistability is a nonlinear characteristic where a specific set of parameters can be associated with more than one stable solution. In these cases, initial conditions define the system behavior.

Bifurcation analysis is useful to identify dynamical system qualitative changes, defining the routes to chaos. In general, two types of bifurcation can be imagined: local and global. Local bifurcations are restricted to regions of phase space. On the other hand, global bifurcations are non-local. Local bifurcation analysis is usually developed based on *normal forms* that represent prototypes of bifurcations (Strogatz, 1994; Wiggins, 1990; Guckenheimer & Holmes, 1983; Savi, 2017).

The local bifurcation analysis can be performed around a bifurcation point, defining possibilities related to dynamical changes. Fig. 13 shows some classical forms of bifurcations related to the creation and destruction of solutions or equilibrium points. It is noticeable the ustable and stable aspects of each solution, which essentially defines the system response.

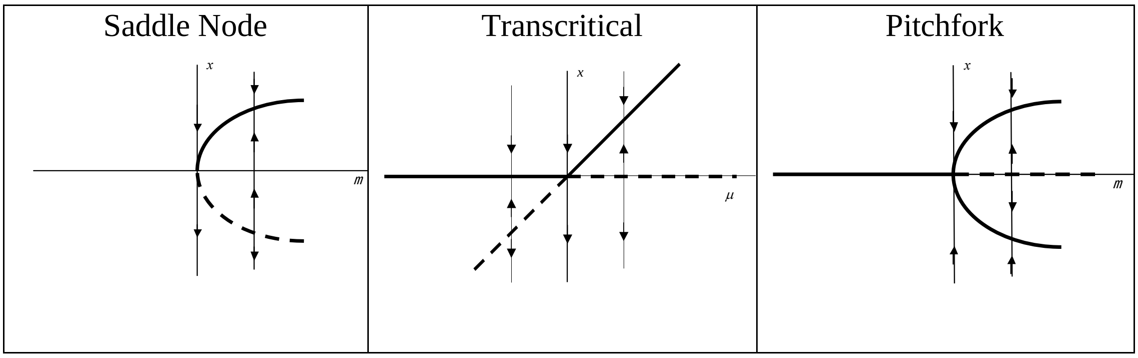


Fig. 13 Examples of local bifurcations observed in dynamical systems.

Global bifurcations are related to qualitative changes in global system aspects and cannot be observed from local analysis. In essence, a parameter change can cause a global change in the orbit structure. This type of bifurcation can explain the birth of chaos due to some orbit collision, for instance. Horseshoe transformation starts due to the transversal touch between unstable and stable manifolds. There is a specific set of parameters that causes this situation, which is responsible for the chaos birth. Fig. 14 shows unstable and stable manifolds for two different set of parameters. Note that the parameter variation promotes a touch of these manifolds, causing the birth of horseshoe transformation and, consequently, the chaos. For more details, see Hirsch *et al.* (2004); Strogatz (1994); Wiggins (1990); Guckenheimer & Holmes (1983); Savi (2017).

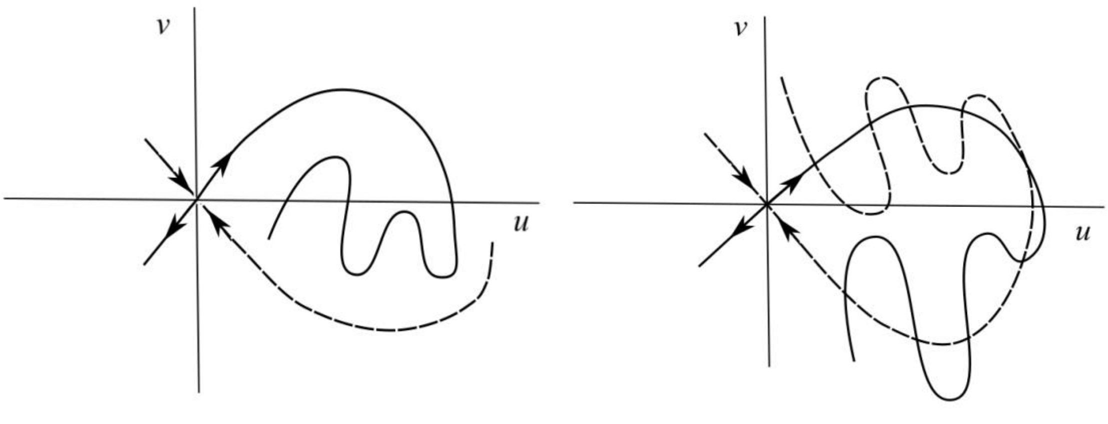


Fig. 14 Global bifurcation represented by two different set of parameters.

Bifurcation diagrams constitute an important tool to identify the influence of parameter changes in system response. It represents a stroboscopically sample of a system variable under the slow quasi-static change of a system parameter. This can be built by a brute force method, just simulating solutions for different parameters. It is important to discard transient response and to define the initial conditions for each parameter. In general, there are two possibilities for the initial conditions: reset for each parameter; use the last simulation as initial conditions. A typical bifurcation diagram is presented in Fig. 15 (Savi, 2017). Unstable solutions are not captured by brute force integrations, needing proper algorithms for that (Parker & Chua, 1989; Alligood *et al.*, 1997).

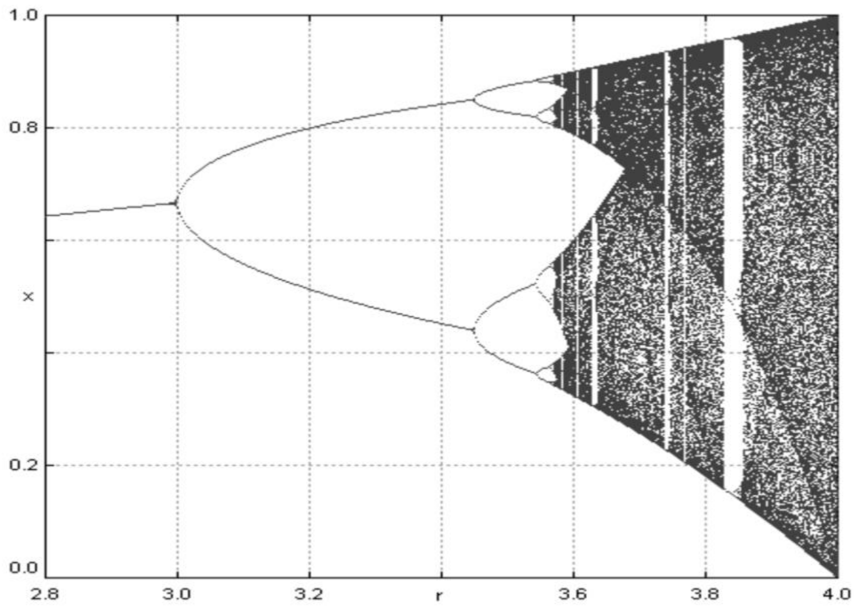


Fig. 15 Bifurcation digram.

5 Lyapunov Exponents

Chaotic behavior needs to be properly identified and diagnostic tools are essential for this aim. Attractor dimension and Lyapunov exponents are dynamical system invariants usually employed to identify chaos.

Lyapunov exponents evaluate the sensitive dependence on initial conditions estimating the local divergence of nearby orbits. These exponents have been used as the most useful diagnostic tool for chaotic system analysis and can also be used for the calculation of other invariant quantities as the attractor dimension.

In order to understand the idea related to the determination of Lyapunov exponents consider a *D*-sphere of states that is transformed by the system dynamics in a

D-ellipsoid. Lyapunov exponents are related to the contraction-expansion nature of different directions in phase space. The evaluation of the divergence of two nearby orbits is done considering the relation between the initial *D*-sphere and the *D*-ellipsoid (Fig. 16). It should be highlighted that, as a matter of fact, there is a spectrum of Lyapunov exponents representing all system dimensions. Based on that, these exponents are expressed by:

$$d(t) = d_0 b^{\lambda t} \tag{5}$$

where d is the lenght, b is a reference basis, and λ is the Lyapunov exponent. Hence, there is a Lyapunov spectrum given by,

$$\lambda = \frac{1}{t}\log_b\left(\frac{d(t)}{d_0}\right) \tag{6}$$

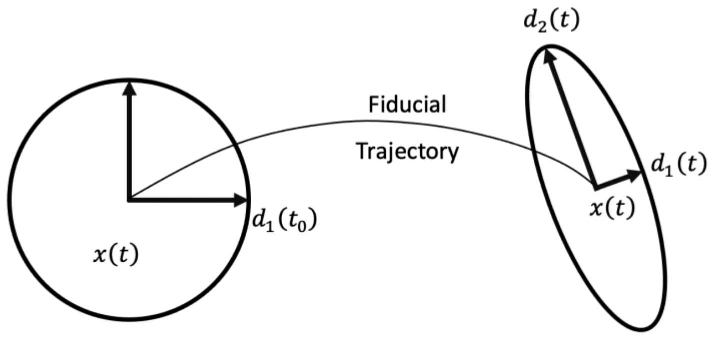


Fig. 16 Lyapunov exponents.

The signs of Lyapunov exponents provide a qualitative picture of the system dynamics. The existence of positive Lyapunov exponents defines directions of local instabilities and any system containing at least one positive exponent presents chaotic behavior. A response with more than one positive exponent is called hyperchaos (Savi & Pacheco, 2002; Franca & Savi, 2003). Negative or null Lyapunov exponent are associated with trajectories that do not diverge. In addition to the signs of the Lyapunov exponents, their values also bring important information related to system dynamics.

Since the exponents evaluate the average divergence of nearby orbits, dissipative systems have a negative sum of the whole Lyapunov spectrum.

The determination of Lyapunov exponents of dynamical system with an explicit mathematical model, which can be linearized, is well established from the algorithm proposed by Wolf *et al.* (1985). On the other hand, the determination of these exponents from time series is quite more complex. In essence, there are two different classes of algorithms: trajectories, real space or direct method; and perturbation, tangent space or Jacobian matrix method (Wolf *et al.*, 1985; Kantz & Schreiber, 1997; Franca & Savi, 2003; Savi, 2017).

Since chaotic situations are related to local exponential divergence of nearby orbits, it is necessary proper algorithms in order to evaluate Lyapunov exponents (Wolf *et al.*, 1985; Parker & Chua, 1989). These algorithms evaluate the average of this divergence considered in different points of the trajectory. Hence, when the distance $d(t)$ becomes large, it is defined a new $d_0(t)$ in order to evaluate the divergence, as follows (Fig. 17):

$$\lambda = \frac{1}{t_n - t_0} \sum_{k=1}^{n} \log_b \left(\frac{d(t_k)}{d_0(t_{k-1})} \right) \tag{7}$$

The new value for the perturbed orbit is defined from de *Gram-Schmidt normalization*. Besides that, the perturbed system can be monitored by different ways (Barreto Netto *et al.*, 2020): *linearized system*, evaluated from the the Jacobian matrix of the system (Wolf *et al.*, 1985); *cloned dynamics*, evaluated by a clone of the the governing equations (Soriano *et al.*, 2012).

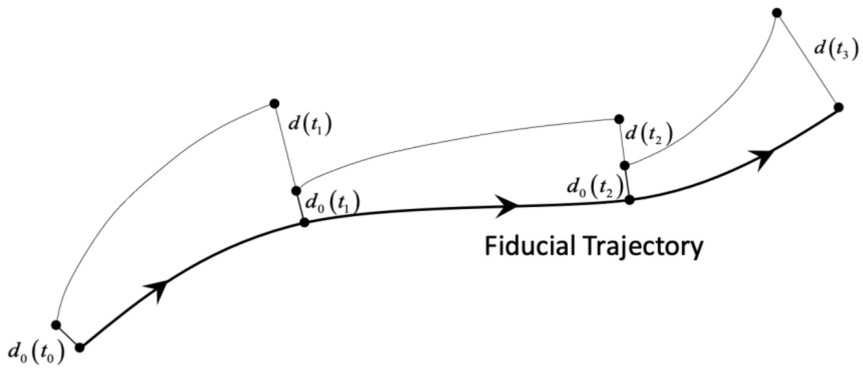


Fig. 17 Lyapunov exponent estimation.

Lyapunov exponents can be employed to calculate other system invariants as attractor dimension. The Kaplan-Yorke conjecture establishes a way to calculate attractor dimension from the spectrum of Lyapunov exponents (Savi, 2017):

$$D = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|} \tag{8}$$

where j is defined as follows,

$$\sum_{i=1}^{j} \lambda_i > 0 \quad \text{and} \quad \sum_{i=1}^{j+1} \lambda_i < 0 \tag{9}$$

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References

Alligood, K.T., Sauer, T.D. & Yorke, J.A. (1997), "*Chaos: An Introduction to Dynamical Systems*", Springer-Verlag.

Barreto Netto, D.M., Brandão, A., Paiva, A., Pacheco, P.M.C.L. & Savi, M.A. (2020), "Estimating Lyapunov Spectrum on Shape Memory Alloy Oscillators Considering Cloned Dynamics and Tangent Map Methods", *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, v.42, n.9, Article 475.

Briggs, J. & Peat, F.D. (2000), "*Seven Life Lessons of Chaos*", Harper Perennial.

Franca, L.F.P. & Savi, M.A. (2003), "Evaluating Noise Sensitivity on the Time Series Determination of Lyapunov Exponents Applied to the Nonlinear Pendulum", *Shock and Vibration*, v.10, n.1, pp.37-50.

Gleick, J. (1997), "*Chaos*", Random House.

Grebogi, C., Ott, E., Pelikan, S. & Yorke, J. (1984), "Strange Attractors that are not Chaotic", *Physica 13D*, pp.261-268.

Guckenheimer, J. & Holmes, P. (1983), "*Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*", Springer-Verlag.

Hilborn, R.C. (1994), "*Chaos and Nonlinear Dynamics*", Oxford Press.

Hirsch, M.W., Smale, S. & Devaney, R.L. (2004), "*Differential Equations, Dynamical Systems & An Introduction to Chaos*", Elsevier.

Kantz, H. & Schreiber, T. (1997), "*Nonlinear Time Series Analysis*", Cambridge University Press

Kapitaniak, T. (1991), "*Chaotic Oscillations in Mechanical Systems*", Manchester.

Lorenz, E. (1995), "*The Essence of Chaos*", University of Washington Press.

Lorenz, E. (1963), "Deterministic Nonperiodic Flow", *Journal of Atmosferic Science*, v.20, pp.130-141.

Moon, F.C. (1992), "*Chaotic and Fractal Dynamics*", John Wiley & Sons.

Mullin, T. (1993), "*The Nature of Chaos*", Oxford Press.

Nayfeh, A.H. & Mook, D.T. (1979), "*Nonlinear Oscillations*", John Wiley & Sons.

Ott, E. (1993), "*Chaos in Dynamical Systems*", Cambridge Press.

Parker, T.S. & Chua, L.O. (1989), "*Practical Numerical Algorithms for Chaotic Systems*", Springer-Verlag.

Savi, M.A. (2014), "*Rhythms of Nature*", Editora E-papers (in Portuguese).

Savi, M.A. (2017), "*Nonlinear Dynamics and Chaos*", Second Edition, Editora E-papers (in Portuguese).

Soriano, D. C., Fazanaro, F. I., Suyama, R., Oliveira, J. R., Attux, R. & Madrid, M. K. (2012), "A Method for Lyapunov Spectrum Estimation Using Cloned Dynamics and its Application to the Discontinuously-excited FitzHugh–Nagumo Model", *Nonlinear Dynamics*, v.67, n.1, pp.413-424.

Stewart, I. (1991), "*Does God Play Dice?*", Jorge Zahar Editor (in Portuguese).

Strogatz, S.H. (1994), "*Nonlinear Dynamics and Chaos*", Perseus.

Thompsom, J.M.T. & Stewart, H.B. (1986), "*Nonlinear Dynamics and Chaos*", John Wiley & Sons, Chichester.

Wiggins, S. (1990), "*Introduction to Applied Nonlinear Dynamical Systems and Chaos*", Springer-Verlag, New York.

Wolf, A., Swift, J.B., Swinney, H.L. and Vastano, J.A. (1985), "Determining Lyapunov Exponents from a Time Series", *Physica 16D*, pp.285-317.