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BIFURCATION CONTROL OF A PARAMETRIC PENDULUM

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In this paper, we apply chaos control methods to modify bifurcations in a parametric pendulum-shaker system. Specifically, the extended time-delayed feedback control method is employed to maintain stable rotational solutions of the system avoiding period doubling bifurcation and bifurcation to chaos. First, the classical chaos control is realized, where some unstable periodic orbits embedded in chaotic attractor are stabilized. Then period doubling bifurcation is prevented in order to extend the frequency range where a period-1 rotating orbit is observed. Finally, bifurcation to chaos is avoided and a stable rotating solution is obtained. In all cases, the continuous method is used for successive control. The bifurcation control method proposed here allows the system to maintain the desired rotational solutions over an extended range of excitation frequency and amplitude.

Keywords: Nonlinear dynamics; chaos; control; parametric pendulum.

1. Introduction

The dynamics of a parametrically excited pendulum has been extensively investigated in the literature. It has been found that this system can generate various types of motion, from simple periodic oscillations to complex chaos. While the majority of the authors analyzed oscillatory motion, i.e. [Xu *et al.*, 2005; Bishop *et al.*, 2005; Clifford & Bishop, 1994, 1996], the rotational solutions received less attention in the past. Nevertheless, a number of papers deal with the rotational solutions of the pendulum under parametric excitation, and they are briefly summarized as follows. Koch and Leven [1985] detected bifurcations of the system where harmonic and subharmonic rotating solutions are born by applying the Melnikovs method. Clifford and Bishop [1995], Garira and Bishop [2003] numerically identified types of rotating solutions

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and classified them. Szemplinska-Stupnicka *et al.* [2000], Szemplinska-Stupnicka and Tyrkiel [2002] numerically investigated global bifurcations and various other aspects of the pendulum dynamics including fractal basin boundaries and coexistence of rotating solutions. Xu *et al.* [2005] and most recently Horton *et al.* [2011] have performed extensive numerical simulations concerning different kinds of rotations, together with oscillations and chaos. Capecchi and Bishop [1990], Xu and Wiercigroch [2007] and Lenci *et al.* [2008] investigated rotating solutions analytically.

Although these rotating solutions are found and studied by different authors, it should be pointed out that they only exist over limited parameters range and there are a lot of bifurcations of the system that destabilize this kind of motion. In this regard, the bifurcation control can be useful in maintaining the rotational solutions.

The idea of controlling bifurcation is based on modifying the considered nonlinear system to achieve some desirable dynamical behavior by constructing and applying a controller [Abed & Fu, 1986, 1987; Wen *et al.*, 2006; Chen *et al.*, 2000, 2003; Xiao & Cao, 2009]. Concerning the rotating solutions, the idea is to maintain stability of this motion avoiding either period doubling bifurcation or bifurcation to chaos.

Chaos control exploits the richness of chaotic behavior and may be understood as the use of perturbations for the stabilization of an Unstable Periodic Orbit (UPO) embedded in a chaotic attractor. Chaos control methods are generally divided into discrete and continuous techniques. The continuous methods are exemplified by the so-called delayed feedback control, proposed by Pyragas [1992], which states that chaotic systems can be stabilized by a feedback perturbation proportional to the difference between the present and a delayed state of the system. Numerous research efforts have been dedicated since then to overcome some limitations of the original technique. Pyragas [2006] presents a review concerning improvements and applications of time-delayed feedback methods.

In this work, continuous chaos control methods are employed in order to maintain rotating solutions of the pendulum system by stabilizing unstable periodic orbits of the system. Hence, the main goal here is to avoid bifurcations that destabilize the rotating motion.

The interest in these rotational solutions is motivated by an idea of energy harvesting from the sea waves using a pendulum system. The concept consists in converting the base oscillations of the structure into a rotational motion of the pendulum mass as proposed by Wiercigroch [2003]. In such case, the oscillations of the structure are caused by the sea waves, whereas the pendulum rotational motion provides a driving torque for an electrical generator [Xu, 2005; Horton, 2009; Horton & Wiercigroch, 2008]. In order to explore potentials of this concept, the dynamics of the pendulum system has to be carefully considered and the means of maintaining the periodic rotational solutions have to be developed.

The paper is organized as follows. In the next section continuous chaos control methods are introduced. Then the studied system is presented in Sec. 3, and in Sec. 4, chaos control methods are employed aiming to stabilize unstable periodic orbits embedded into the chaotic attractors, or extend the stability range of the existing periodic orbits by avoiding period doubling bifurcations and bifurcations to chaos.

2. Chaos Control Methods

Chaos control may be understood as the use of small perturbations in order to stabilize unstable periodic orbits. Typically, it is a two-stage procedure including learning stage and control stage. Since unstable periodic orbits are part of the system dynamics, the stabilization of these orbits is associated with low energy consumption. Therefore, this procedure is useful for different applications being of special interest in order to design flexible systems.

Chaos control methods can be split into discrete and continuous methods. Continuous methods are based on continuous-time perturbations to perform stabilization. This approach was first proposed by Pyragas [1992] and deals with a dynamical system modeled by a set of ordinary nonlinear differential equations as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{Q}(\mathbf{x}, t) + \mathbf{B}(t), \qquad (1)$$

where t is time, $\mathbf{x}(t) \in \Re^n$ is the state variable vector, $\mathbf{Q}(\mathbf{x}, t) \in \Re^n$ defines the system dynamics, while $\mathbf{B}(t) \in \Re^n$ is associated with the control action.

Socolar *et al.* [1994] proposed a control law named as the Extended Time-Delayed Feedback control (ETDF) considering the information of time-delayed states of the system represented by the following equations:

$$\mathbf{B}(t) = \mathbf{K}[(1-R)\mathbf{S}_\tau - \mathbf{x}],$$
$$\mathbf{S}_\tau = \sum_{m=1}^{N_\tau} R^{m-1}\mathbf{x}_{m\tau}, \tag{2}$$

where $\mathbf{K} \in \Re^{n\times n}$ is the feedback gain matrix, $0 \leq R < 1$ is a control gain, $\mathbf{S}_\tau = \mathbf{S}(t-\tau)$ and $\mathbf{x}_{m\tau} = \mathbf{x}(t - m\tau)$ are related to delayed states of the system and τ is the time delay. In general, N_τ is infinity but it can be more precisely defined depending on the dynamical system. The UPO stabilization can be achieved by an appropriate choice of $\mathbf{K}$ and R. Note that for any gain defined by $\mathbf{K}$ and R, the perturbation of Eq. (2) vanishes when the system is on the UPO since $\mathbf{x}(t - m\tau) = \mathbf{x}(t)$ for all m if $\tau = T_i$, where $= T_i$ is the periodicity of the ith UPO. It should be pointed out here that when $R = 0$, the ETDF turns into the original Time-Delayed Feedback control method (TDF) proposed by Pyragas [1992].

The controlled dynamical system consists of a set of Delay Differential Equations (DDEs). The solution of this system is done by establishing an initial function $\mathbf{x}_0 = \mathbf{x}_0(t)$ over the interval $(-N_\tau\tau, 0)$. This function can be estimated by a Taylor series expansion as proposed by Cunningham [1954]:

$$\mathbf{x}_{m\tau} = \mathbf{x} - m\tau\dot{\mathbf{x}}. \tag{3}$$

Under this assumption, the following system is obtained:

$$\dot{\mathbf{x}} = \mathbf{Q}(\mathbf{x}, t) + \mathbf{K}[(1-R)\mathbf{S}_\tau - \mathbf{x}],$$

where

$$\mathbf{S}_\tau = \begin{cases} \displaystyle\sum_{m=1}^{N_\tau} R^{m-1}[\mathbf{x} - m\tau\dot{\mathbf{x}}], & \text{for } (t - N_\tau\tau) < 0, \\ \displaystyle\sum_{m=1}^{N_\tau} R^{m-1}\mathbf{x}_{m\tau}, & \text{for } (t - N_\tau\tau) \geq 0. \end{cases} \tag{4}$$

Note that DDEs contain derivatives that depend on the solution at delayed time instants. Therefore, besides the special treatment that must be given for $(t - N_\tau\tau) < 0$, it is necessary to deal with the time-delayed states while integrating the system. A fourth-order Runge–Kutta method with linear interpolation on the delayed variables is employed in this work for the numerical integration of the controlled dynamical system [Mensour & Longtin, 1997]. It is important to note that the Taylor series expansion is used only at the beginning of the integration, while $(t-N_\tau\tau) < 0$. This procedure has been tested by considering different number of terms and results are qualitatively the same. An alternative approach would be to adopt the start of the control action only after all necessary delayed states are known, i.e. when $t > N_\tau\tau$.

Before the control stage, where the desired UPOs are stabilized, it is necessary to identify the UPOs embedded in chaotic attractor, which is achieved by applying the close-return method [Auerbach *et al.*, 1987] and to define controller parameters, $\mathbf{K}$ and R, for each one of the desired orbits. The controlled parameter values for each UPO are defined by calculating the Lyapunov exponents of the corresponding orbit in such a way that the exponents become all negatives and the UPO becomes stable.

2.1. *UPO Lyapunov exponents*

The idea behind the delayed feedback control is the construction of a continuous-time perturbation, as presented in Eq. (2), proposed by Kittel *et al.* [1995] or by Pyragas [1993], which does not change the desired UPO of the system, but only modifies its characteristics. This is done by changing the control parameters in such a way that Lyapunov exponents become all negatives and the UPO becomes stable [Kittel *et al.*, 1995]. To perform this analysis, it is sufficient to determine only the largest Lyapunov exponent in order to find values of $\mathbf{K}$ and R where the control is achieved. In other words, it is necessary to look for a situation where the maximum exponent is negative, $\lambda(\mathbf{K}, R) < 0$. Besides, Pyragas [1995] states that the minimum of $\lambda(\mathbf{K}, R)$ provides a faster convergence rate of nearby orbits to the desired UPO and makes the method more robust with respect to noise.

The estimation of Lyapunov exponent of DDEs is more complicated than that of ODEs. This happens because the terms associated with the extended delayed feedback, Eq. (2), involves the knowledge of the system states delayed in time. By considering three delayed states, for example, Eq. (1) becomes a system composed of delay differential equations as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{Q}(\mathbf{x}, t) + \mathbf{B}(t, \mathbf{x}, \mathbf{x}_\tau, \mathbf{x}_{2\tau}, \mathbf{x}_{3\tau}). \tag{5}$$

Therefore, calculations of $\mathbf{x} = \mathbf{x}(t)$ imply that functions $x_i(t), i = 1, \ldots, n$, over the interval $[t - 3\tau, t)$ must be known, where n corresponds to the system dimension without a control law. Equation of this type describes an infinite-dimensional system, and it should present an infinite number of Lyapunov exponents, from which only a finite portion of them can be determined by numerical analysis [Vicente *et al.*, 2005].

For the stability analysis of the UPO, however, considering nonautonomous systems, it is enough to determine only the largest Lyapunov exponent since these systems do not have a zero exponent associated with the tangent to the flow direction [Pyragas, 1995].

In this work, the calculation of Lyapunov exponents is conducted by approximating the continuous evolution of the infinite-dimension system by a finite number of elements whose values change at discrete time steps [Farmer, 1982]. In this regard, the function $x_i(t), i = 1, \ldots, n$, over the interval $[t - 3\tau, t - \Delta t]$ can be approximated by N samples taken at intervals Δt, which corresponds to the integration time step. Therefore, instead of n continuous time dependent variables presented in Eq. (5), $n(N + 1)$ variables are now considered and represented by vector $\mathbf{z}$, where components $z_1(t), \ldots, z_n(t)$ are the original functions of time $x_1(t), \ldots, x_n(t)$ and components $z_{n+1}, \ldots, z_{n(N+1)}$ are unknown values of the functions $x_i(t)$ at specific times $t - \Delta t, t - 2\Delta t, \ldots, t - N\Delta t$ related to delayed states of $\mathbf{x}(t)$:

$$\begin{aligned}(&z_1, z_2, \ldots, z_n, z_{n+1}, \ldots, z_{n+N}, \ldots,\\ &\quad z_{n+(n-1)N+1}, \ldots, z_{n(N+1)})\\ &= (x_1(t), x_2(t), \ldots, x_n(t), x_1(t - \Delta t), \ldots,\\ &\quad x_1(t - N\Delta t), \ldots, x_n(t - \Delta t), \ldots,\\ &\quad x_n(t - N\Delta t)). \end{aligned} \tag{6}$$

There are different ways to perform this type of approximation. In this work, based on the procedure used by Sprott [2007], the DDE is replaced by a set of ODEs. Under this assumption, the continuous infinite-dimensional system, Eq. (5), is represented in terms of $n(N + 1)$ ODEs, where special equations are used to approximate the derivatives of the unknown values $z_{n+1}, \ldots, z_{n+N}, \ldots, z_{n+(n-1)N+1}, \ldots, z_{n(N+1)}$ which in this way also become functions of time:

$$\begin{aligned}
\dot{z}_j &= \mathbf{Q}_j(z_1, z_2, \ldots, z_n)\\
&\quad + \mathbf{B}_j(t, z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n(N+1)}), && \text{for } 1 \le j \le n\\
\dot{z}_{n+1+(j-1)N} &= \frac{N(z_j - z_{n+2+(j-1)N})}{2T}, && \text{for } 1 \le j \le n\\
\dot{z}_{n+i+(j-1)N} &= \frac{N(z_{n+i+(j-1)N-1} - z_{n+i+(j-1)N+1})}{2T}, && \text{for } 2 \le i \le (N-1) \quad \text{and} \quad 1 \le j \le n\\
\dot{z}_{n+jN} &= \frac{N(z_{n+jN-1} - z_{n+jN})}{T}, && \text{for } 1 \le j \le n
\end{aligned} \tag{7}$$

where $N = 3\tau/\Delta t + 1$. This system can be solved by any standard integration methods such as the fourth-order Runge–Kutta. With the representation presented in Eq. (7), the Lyapunov exponents can be calculated by using the algorithm proposed by Wolf *et al.* [1985]. Moreover, in order to calculate the Lyapunov exponent of a specific UPO, the system is integrated along the desired orbit.

3. Parametric Pendulum-Shaker System

The original idea that energy harvesting can be provided from sea waves proposed by Wiercigroch [2003] was investigated in [Xu, 2005; Horton, 2009; Horton & Wiercigroch, 2008] looking at the dynamics of a parametrically driven pendulum. Motivated by this idea a follow up work was conducted by the same group in [Xu *et al.*, 2007; Horton *et al.*, 2008] where the behavior of a parametric pendulum excited by an electro-dynamical shaker was analyzed and which also has been chosen to be studied in this work. Figure 1 presents a schematic picture of this system identifying mechanical and electrical components. The mechanical system [Fig. 1(a)] is comprised of three masses: the pendulum mass, M, the armature assembly mass, M_a, and the body mass, M_b, that represents the mass of the magnetic structure containing the field coil. The excitation is

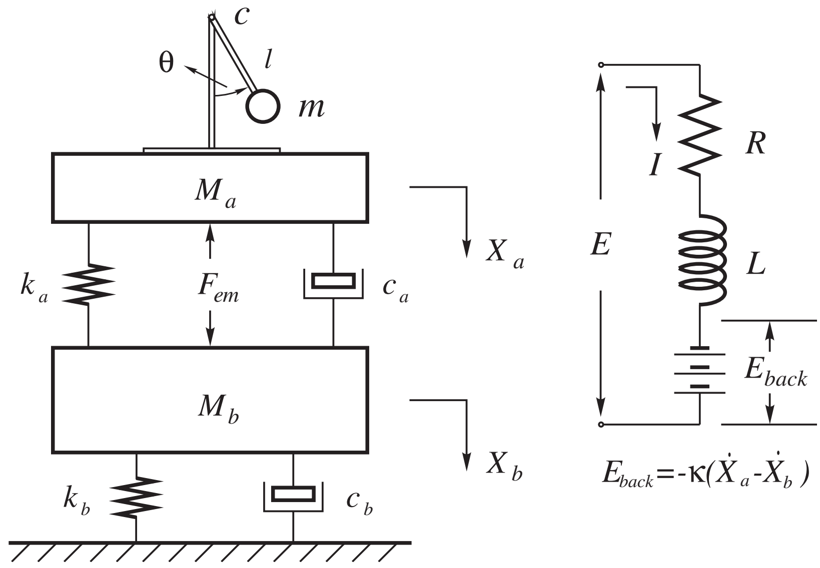


Fig. 1. Physical model of the pendulum-shaker system with mechanical and electrical components. Adopted from [Xu *et al.*, 2007].

provided by an axial electromagnetic force, F_{em}, which is generated by the alternating current in the constant magnetic field.

The mechanical part of the pendulum-shaker system is described by three generalized coordinates: angular displacement of the pendulum, θ, and the vertical displacements of the body and the armature, X_b and X_a, respectively. The electrical system is described by the electric charge q, that is related to the current I by its derivative: $I = dq/dt$. Equations of motion for each degree-of-freedom of the parametric pendulum-shaker system are given by:

$$\begin{aligned}
\theta &: \quad Ml\ddot{\theta} + cl\dot{\theta} + Mg\sin\theta = M\ddot{X}_a\sin\theta + \frac{T_C}{l},\\
X_a &: \quad (M_a + M)\ddot{X}_a + c_a(\dot{X}_a - \dot{X}_b) + k_a(X_a - X_b) = (M_a + M)g + Ml\ddot{\theta}\sin\theta + Ml\dot{\theta}^2\cos\theta - \kappa I,\\
X_b &: \quad M_b\ddot{X}_b + c_b\dot{X}_b - c_a(\dot{X}_a - \dot{X}_b) + k_bX_b - k_a(X_a - X_b) = M_bg + \kappa I,\\
I &: \quad R_EI + L\frac{dI}{dt} - \kappa(\dot{X}_a - \dot{X}_b) = E_0\cos\Omega t,
\end{aligned} \tag{8}$$

where T_C corresponds to the control parameter actuation, which consists of a torque applied to the pendulum. Considering the state variables $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\} = \{\theta, \dot{\theta}, X_a, \dot{X}_a, X_b, \dot{X}_b, I\}$, the equations of motion are now written as a set of first-order differential equations:

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = \frac{\left(\frac{T_C}{l} - clx_2\right)(M_a + M) - [c_a(x_4 - x_6) + k_a(x_3 - x_5) + \kappa x_7]M\sin x_1 + M^2lx_2^2\cos x_1\sin x_1}{Ml(M_a + M - M\sin^2 x_1)},$$

$$\dot{x}_3 = x_4,$$

$$\dot{x}_4 = \frac{(M_a + M)g + Mlx_2^2\cos x_1 - \kappa x_7 - c_a(x_4 - x_6) - k_a(x_3 - x_5) - clx_2\sin x_1 - mg\sin^2 x_1}{M_a + M - M\sin^2 x_1},$$

$$\dot{x}_5 = x_6,$$

$$\dot{x}_6 = \frac{M_b g + \kappa x_7 - c_6 x_6 + c_a(x_4 - x_6) - k_b x_5 + k_a(x_3 - x_5)}{M_b},$$

$$\dot{x}_7 = \frac{E_0 \cos(\Omega t) - R_E x_7 + \kappa(x_4 - x_6)}{L}. \tag{9}$$

By using the formalism presented for the extended time-delayed feedback control law, the control actuation T_C, with $N_\tau = 3$, may be expressed as follows:

$$T_C = \frac{Ml^2(M_a + M - M\sin^2 x_1)}{(M_a + M)} \times K[(1 - R)(x_\tau + Rx_{2\tau} + R^2 x_{3\tau}) - x_2], \tag{10}$$

where $x_\tau = x_2(t - \tau)$, $x_{2\tau} = x_2(t - 2\tau)$ and $x_{3\tau} = x_2(t-3\tau)$. Moreover, matrix gain $\mathbf{K}$ becomes a scalar K once the control action is only applied to one differential equation, the one related to time evolution of x_2, and its control law is only associated with delayed values of one state variable, x_2. In other words, only component K_{22} is different from zero and notation K is used for it. Therefore, in this case only N new variables will be required to approximate the values of $x_2(t)$ at the delayed states as presented in Eq. (11).

$$\dot{z}_8 = \frac{N(z_2 - z_9)}{2\tau}$$

$$\dot{z}_j = \frac{N(z_{i-1} - z_{i+1})}{2\tau}, \quad \text{if } 9 \le i < N + 7 \tag{11}$$

$$\dot{z}_{N+7} = \frac{N(z_{N+6} - z_{N+7})}{\tau}$$

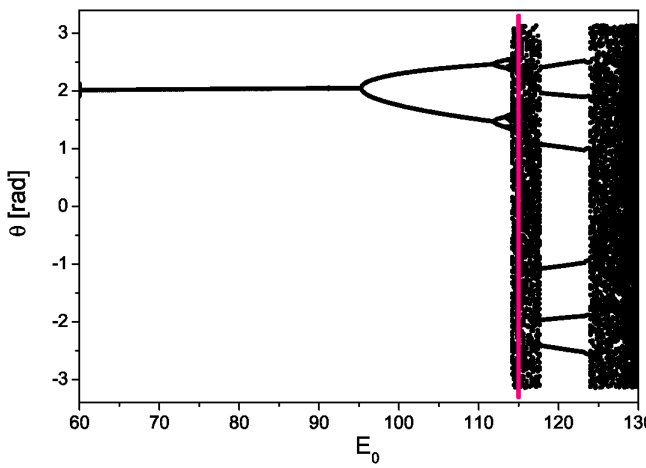
where $z_8(t), \ldots, z_{N+7}(t)$ correspond to delayed states of $x_2(t)$ and $z_1(t), \ldots, z_7(t)$ correspond to $x_1(t), \ldots, x_7(t)$.

Xu *et al.* [2007] discussed experimental aspects of the pendulum-shaker dynamics. Here, we use the proposed model with experimentally determined parameters presented in Table 1.

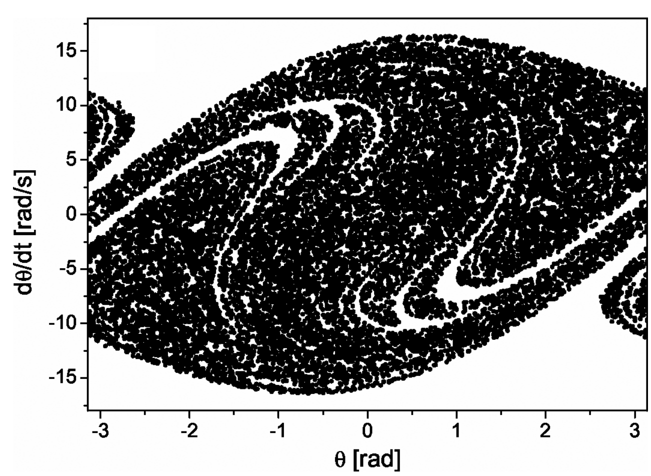
Table 1. Experimentally determined parameters of the pendulum-shaker system [Xu *et al.*, 2007].

M	0.845 kg	l	0.3166 m	c	0.0475 kg/s
M_a	68.38 kg	k_a	86175.9 kg/s^2	c_a	534.05 kg/s
M_b	820 kg	k_b	244284 kg/s^2	c_b	679.35 kg/s
R_E	0.3 Ω	L	2.626×10^{-3} H	κ	130 N/A

At this point, let us briefly analyze the uncontrolled system behavior. Bifurcation diagrams are constructed by assuming a quasi-static stroboscopic increase of the voltage amplitude E_0 with initial value of $E_0 = 60$ V. The first 200 periods are neglected in order to reach the steady state response. The diagram in Fig. 2(a) shows period doubling bifurcations that lead to chaotic regions and then, periodic window with a period-6 response. It is calculated for $\Omega = 9$ rad/s, whereas a Poincaré



(a)



(b)

Fig. 2. (a) Bifurcation diagram constructed at $\Omega = 9$ rad/s and (b) Poincaré section at $E_0 = 115$ V.

section of the chaotic attractor is shown in Fig. 2(b) for $E_0 = 115\,\mathrm{V}$ [marked by pink line in Fig. 2(a)] considering the phase space bounded within $(-\pi, +\pi)$.

In order to explore some details of the dynamical behavior of the pendulum-shaker system, a different bifurcation diagram is now constructed under the variation of frequency parameter Ω and $E_0 = 85\,\mathrm{V}$. Figure 3 presents three different situations: increasing the forcing frequency, in pink, and decreasing the frequency with different initial conditions, in black and in gray. The analysis of Fig. 3 shows that the system seems to have similar behavior at $\Omega = 12.2\,\mathrm{rad/s}$ and $\Omega = 10.25\,\mathrm{rad/s}$.

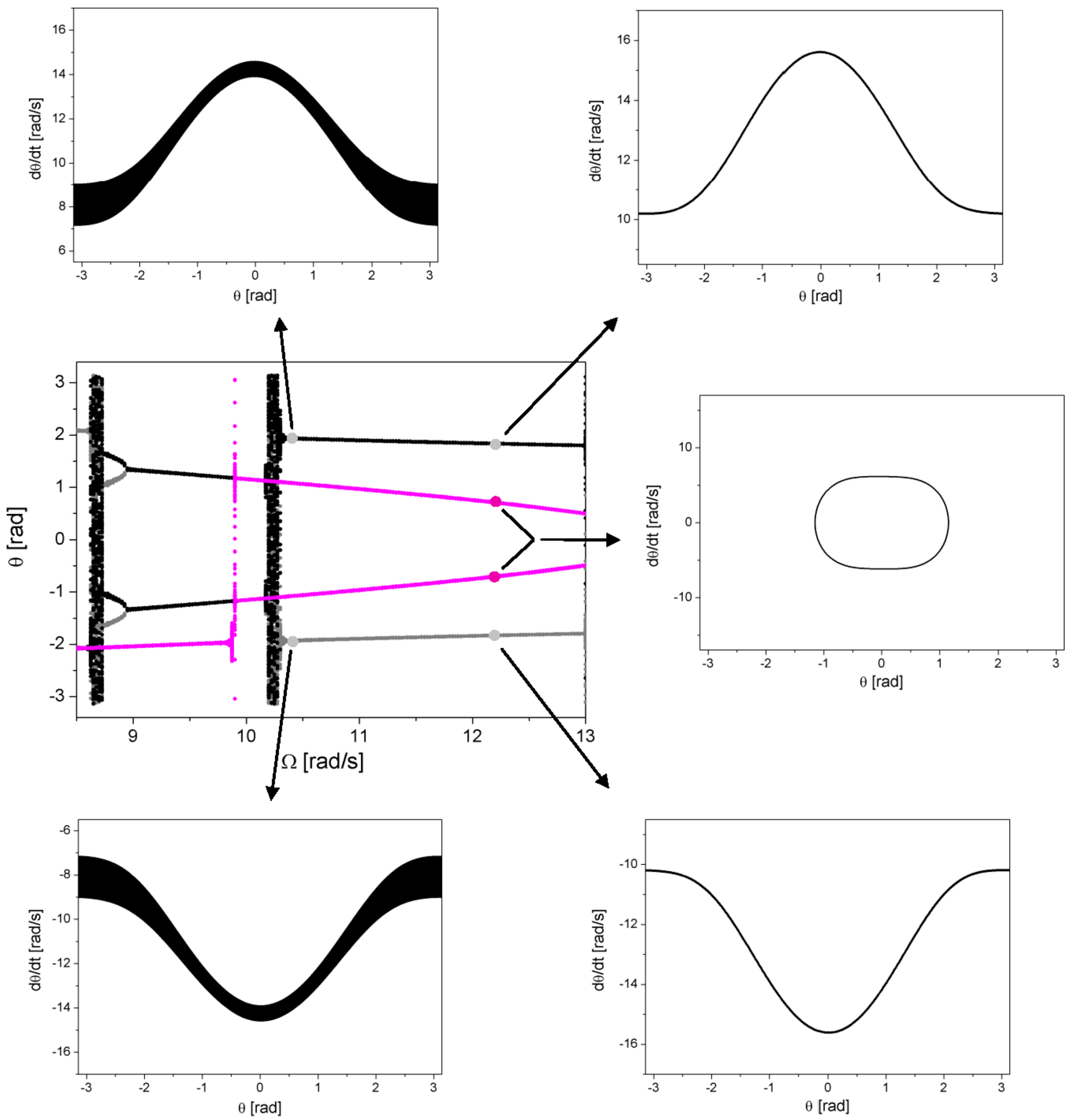


Fig. 3. Bifurcation diagram at $E_0 = 85\,\mathrm{V}$ calculated by increasing and decreasing the forcing frequency. Different coexisting solutions are highlighted.

Nevertheless, in the first case the system presents three coexisting periodic attractors, while at $\Omega = 10.25\,\text{rad/s}$, there are two quasi-periodic and one periodic attractors coexisting. The phase space of the coexisting orbits at $\Omega = 12.2\,\text{rad/s}$, two period-1 and one period-2, and at $\Omega = 10.25\,\text{rad/s}$, two quasi-periodic and one period-2, are depicted in Fig. 3.

The analysis of the system dynamics shows that the parametric pendulum may present different kinds of solutions. Periodic, quasi-periodic and chaotic responses are all present in the system behavior. Besides, it is important to highlight the coexisting attractors that can occur for the same set of parameters. Therefore, bifurcation control is important when we are thinking of the use of this pendulum system in some application. In terms of energy harvesting, it is of special interest to keep a period-1 rotating orbit, avoiding any kind of bifurcation. Another interesting procedure could be the stabilization of a rotation solution immersed in chaotic attractor. In this regard, we will apply control techniques for bifurcation control.

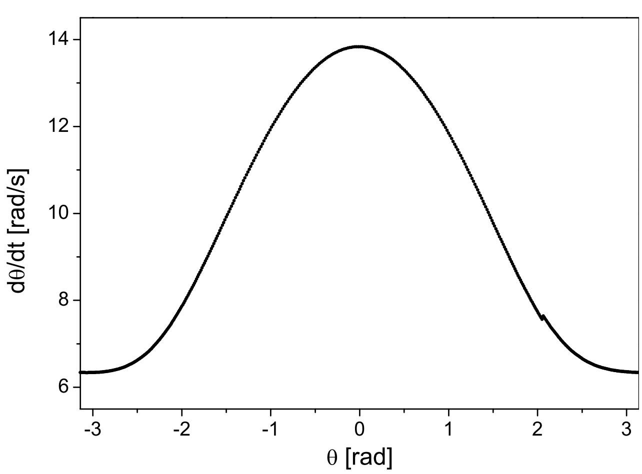
4. Controlling the Pendulum-Shaker System

In this section, continuous chaos control methods are employed in order to stabilize unstable periodic orbits. Different scenarios are considered. First, classical chaos control is performed, stabilizing some UPOs embedded in chaotic attractor. Then, period doubling bifurcation is avoided in order to keep a period-1 rotating orbit. Finally, bifurcation to chaos is prevented by stabilizing originally unstable period-1 rotating orbit. All these scenarios are important for energy harvesting purposes.

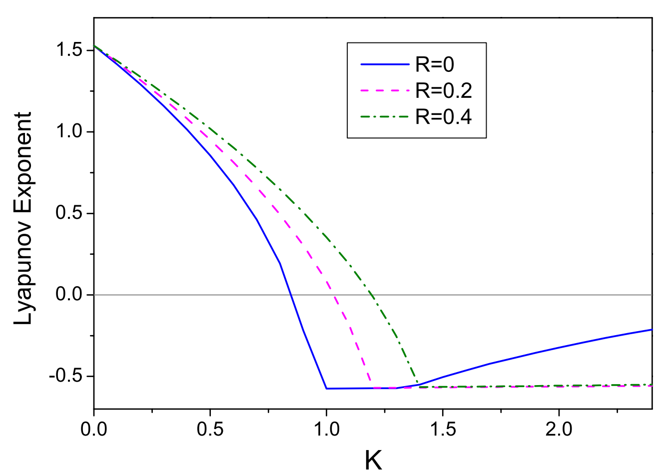
4.1. *Chaos control*

In this subsection, continuous chaos control approach is employed in order to stabilize some UPOs of the pendulum-shaker system. The first step of this control is the learning stage where UPO is identified and control parameters are chosen. Figure 4(a) presents a period-1 UPO identified by the close return method [Auerbach *et al.*, 1987] at $\Omega = 9\,\text{rad/s}$ and $E_0 = 115\,\text{V}$, related to the Poincaré section presented in Fig. 2. Figure 4(b) shows its largest Lyapunov exponents considering different control parameters. It is important to highlight that in Lyapunov exponent calculation, as well as in the control stage, the value of τ is equal to the periodicity of the desired UPO. The analysis of the control parameters shows that there are different alternatives to obtain negative values of Lyapunov exponents and these values can be used for control purposes.

Control is now applied with the purpose to stabilize the period-1 UPO (rotating solution) assuming $R = 0$ and $K = 1.2$. Figure 5 shows pendulum displacement and applied torque evolution in time. The initial function $x_0 = x_0(t)$ over the time interval $[-N_\tau\tau, 0)$ is estimated by the Taylor series expansion presented in Eq. (3) and $x_0(0) = \{-3, 0, \ldots, 0\}$. Figure 6 presents pendulum phase space during the control period highlighting the steady-state response (in pink).

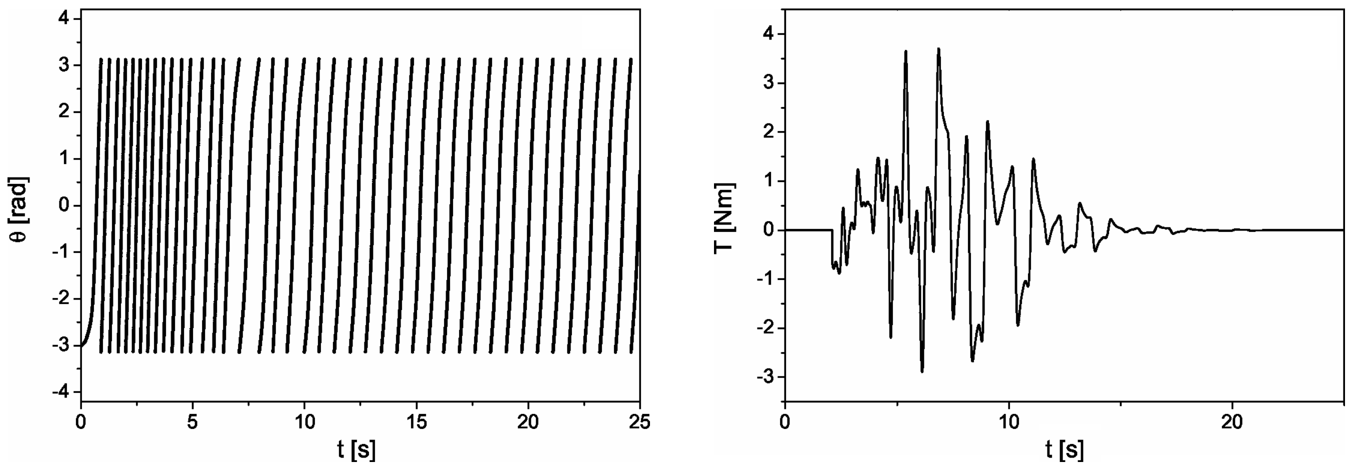


(a)



(b)

Fig. 4. (a) Period-1 UPO, (b) its largest Lyapunov exponent for different values of the control parameters K and R.



(a) (b)

Fig. 5. Period-1 UPO stabilized for $R = 0$, $K = 1.2$ and $x_0(0) = \{-3, 0, \ldots, 0\}$: (a) time response, (b) applied torque.

4.2. *Control to avoid period doubling bifurcation*

If the system is required to operate in a stable rotational mode, the qualitative changes in system response must be avoided over the extended parameter range. The bifurcations of the system can be prevented if chaos control methods are applied. Figure 7(a) shows a period-1 orbit for $E_0 = 95.5\,\text{V}$ and $\Omega = 9\,\text{rad/s}$, before the period doubling bifurcation, while Fig. 7(b) presents the largest Lyapunov exponent of this orbit for different control parameters, $\Omega = 9\,\text{rad/s}$ and $E_0 = 96\,\text{V}$, a value after the period doubling bifurcation. Figure 8 shows the bifurcation diagram of pendulum displacement at $\Omega = 9\,\text{rad/s}$ without control action (black) and with control action (pink) considering two different sets of control parameters: $R = 0$, $K = 0.6$ and $R = 0$, $K = 1$. Note that bifurcation is avoided in both cases but the second case is more effective to maintain the period-1 response.

In this work, the control parameters associated with the minimum value of the largest Lyapunov exponent are chosen to stabilize the system [Pyragas, 1995]. In Fig. 7(b), however, it can be observed that the same minimum value of Lyapunov exponent is obtained for $0.5 < K < 1.1$ and $R = 0$. When the different values of K, with $R = 0$, are used to perform the bifurcation control, the larger values of K lead to a more effective control, which means that stability of the rotational solution is maintained over larger range of E_0. This can be seen from Fig. 8(b), where for $K = 1$, the period-1 rotational orbit remains stable for the whole range of the analyzed E_0. Therefore, in this case the optimal value of K can be chosen depending on the considered voltage amplitude range to avoid unnecessary high control effort.

It is important to mention that the periodic orbits presented in Fig. 8 in pink are obtained by stabilizing the unstable periodic orbits of the original system. Therefore, the variation of the parameter E_0 induces a change in the response which is compensated by the control action. Figure 9 shows the time history of applied torque under the performed perturbations, encompassing three increases of the value of E_0. Note that the variation of the parameter E_0 is related to a transient response of the controller, and after each increase in E_0, a relatively small control action is required to stabilize the desired periodic orbit, but once it is done,

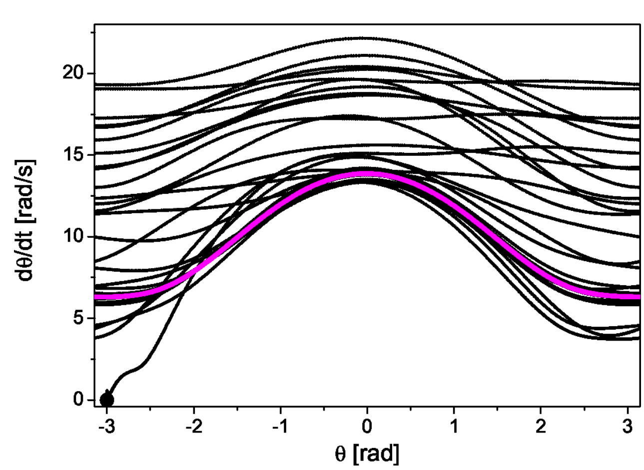


Fig. 6. Phase space controlled response for $R = 0$, $K = 1.2$ and $x_0(0) = \{-3, 0, \ldots, 0\}$. The steady state response is marked in pink.

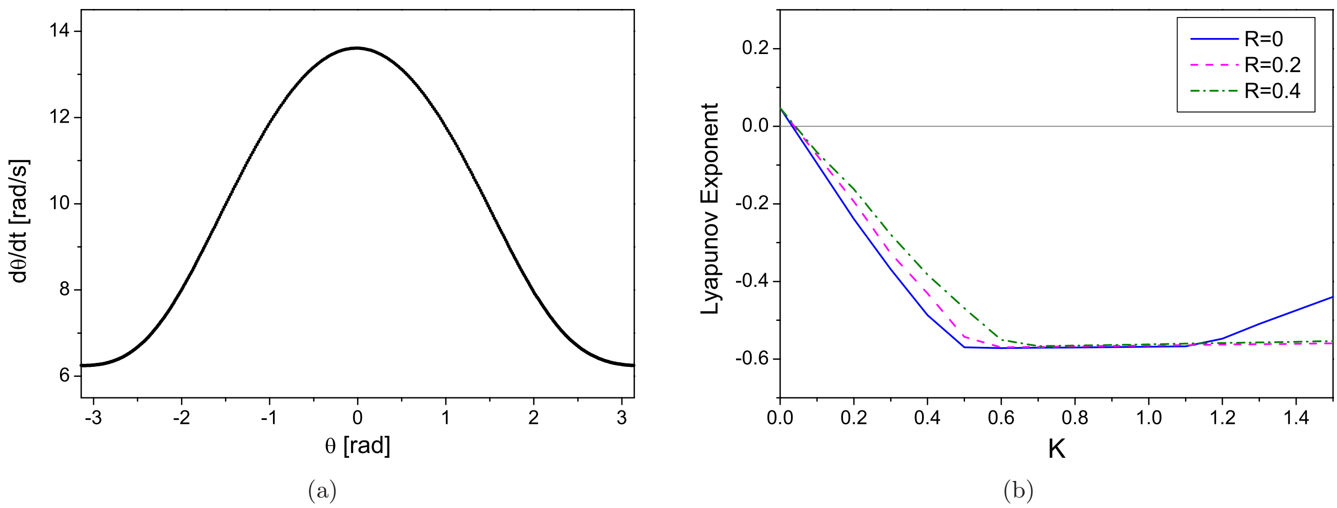


Fig. 7. (a) Period-1 orbit pendulum response for $E_0 = 95.5\,\text{V}$, (b) its largest Lyapunov exponent for $E_0 = 96\,\text{V}$ and different control parameters.

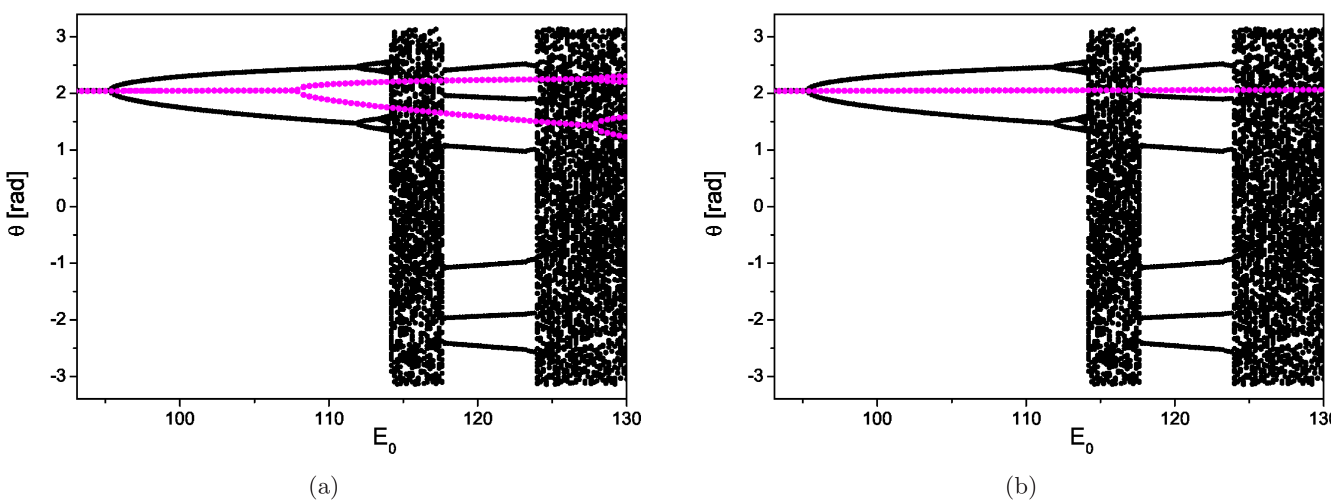


Fig. 8. Bifurcation diagram at $\Omega = 9\,\text{rad/s}$ without control action (black) and with control action (pink): (a) $R = 0$ and $K = 0.6$, (b) $R = 0$ and $K = 1$.

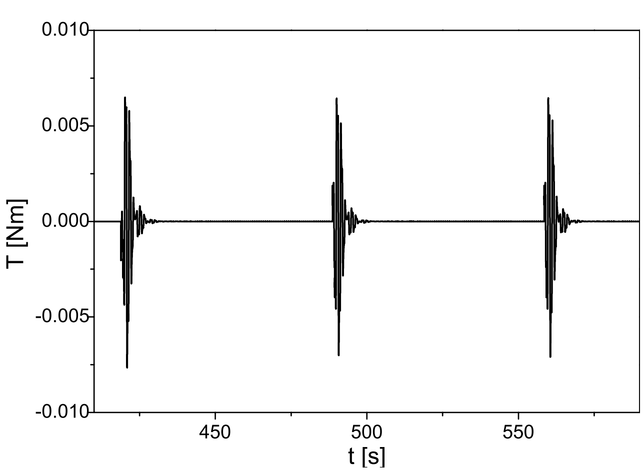


Fig. 9. Detail of the performed control action in bifurcation control.

there is little control effort needed to maintain this regime.

4.3. *Control to avoid bifurcation to chaos — Preserving rotating orbits*

A different possibility concerning the control of the pendulum-shaker system is to avoid the chaotic behavior and to maintain the period-1 rotating response. Figure 10 presents some coexisting orbits identified in the bifurcation diagram of Fig. 3. Figure 10(a) presents a phase space with three orbits: two rotating period-1 orbits and an oscillating period-2 orbit that coexists at $\Omega = 12.2\,\text{rad/s}$ and $E_0 = 85\,\text{V}$. Figure 10(b) presents the phase

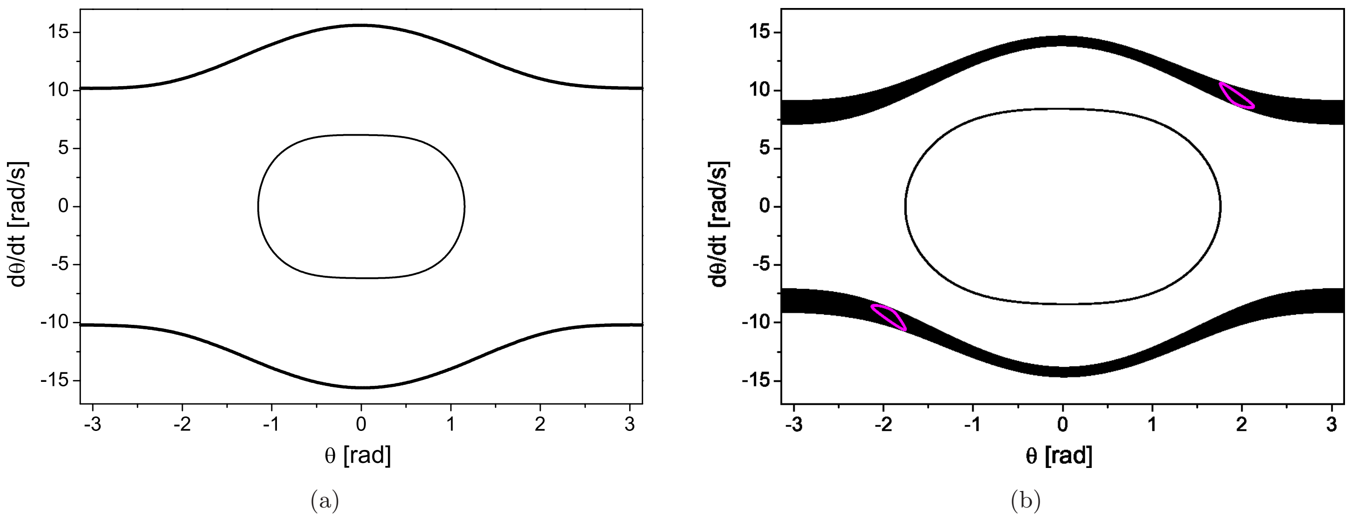


Fig. 10. Phase space of coexisting attractors: (a) three periodic regimes at $E_0 = 85\,\text{V}$ and $\Omega = 12.2\,\text{rad/s}$, (b) one periodic and two quasi-periodic regimes, together with their Poincaré sections, at $E_0 = 85\,\text{V}$ and $\Omega = 10.25\,\text{rad/s}$.

space of the same orbits at $\Omega = 10.25\,\text{rad/s}$ and $E_0 = 85\,\text{V}$ showing the change of the two period-1 rotating orbits for two quasi-periodic rotating orbits. This picture also presents the Poincaré sections of quasi-periodic responses.

The control is now applied to these orbits, and the case where the forcing frequency is decreasing is considered. The aim is to maintain a rotating orbit, avoiding the bifurcation to chaos. Moreover, since the period-2 orbit is not a rotating orbit, this response is not desirable. Decreasing the frequency, the system presents the following route to chaos: period-1 orbit, then quasi-periodic orbit and then chaos. Before the bifurcation to chaos the system presents a quasi-periodic behavior, the period-1 "positive" orbit identified at $E_0 = 85\,\text{V}$ and $\Omega = 12.2\,\text{rad/s}$ is considered as the fiducial trajectory. Figure 11 presents the largest Lyapunov exponent of this orbit for different control parameters at $E_0 = 85\,\text{V}$ and $\Omega = 10.335\,\text{rad/s}$, a set of parameters related to chaotic behavior.

After the learning stage, control is applied to the system. Figure 12 shows the bifurcation diagram constructed at $E_0 = 85\,\text{V}$ with and without control action. Controller uses $R = 0.2$ and $K = 0.4$ with the objective of keep the rotating response of the system. At first, the controller waits until the system trajectory "falls" in the neighborhood of the desired orbit to be in the control on mode [Fig. 12(a)]. In other words, the system is integrated without control action until its trajectory goes into the neighborhood of the desired UPO. When this neighborhood, defined by considering a given tolerance, is reached the control action begins. This wait time is necessary due to the quasi-periodic behavior. If this wait time is not considered and the control action begins in the first integration time step, the controller is not able to stabilize the desired orbit as shown in Fig. 12(b). This wait time is an essential characteristic of discrete chaos control method [Ott *et al.*, 1990; Hübinger *et al.*, 1994; De Paula & Savi, 2008, 2009a] usually not being employed in continuous methods [Pyragas, 1992; Socolar *et al.*, 1994; De Paula & Savi, 2009b].

Figure 13 presents the controlled orbit after the wait time for $E_0 = 85\,\text{V}$ and two different situations: $\Omega = 10.3\,\text{rad/s}$ and $\Omega = 10.15\,\text{rad/s}$. It is clear that

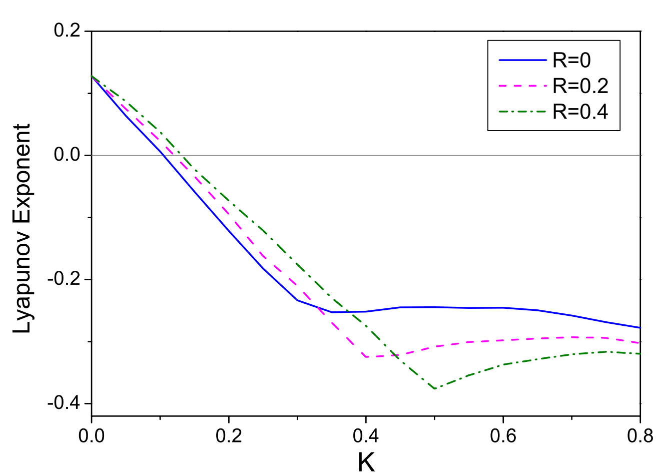


Fig. 11. Largest Lyapunov exponent of the period-1 orbit at $E_0 = 85\,\text{V}$ and $\Omega = 10.335\,\text{rad/s}$ for different values of R and K.

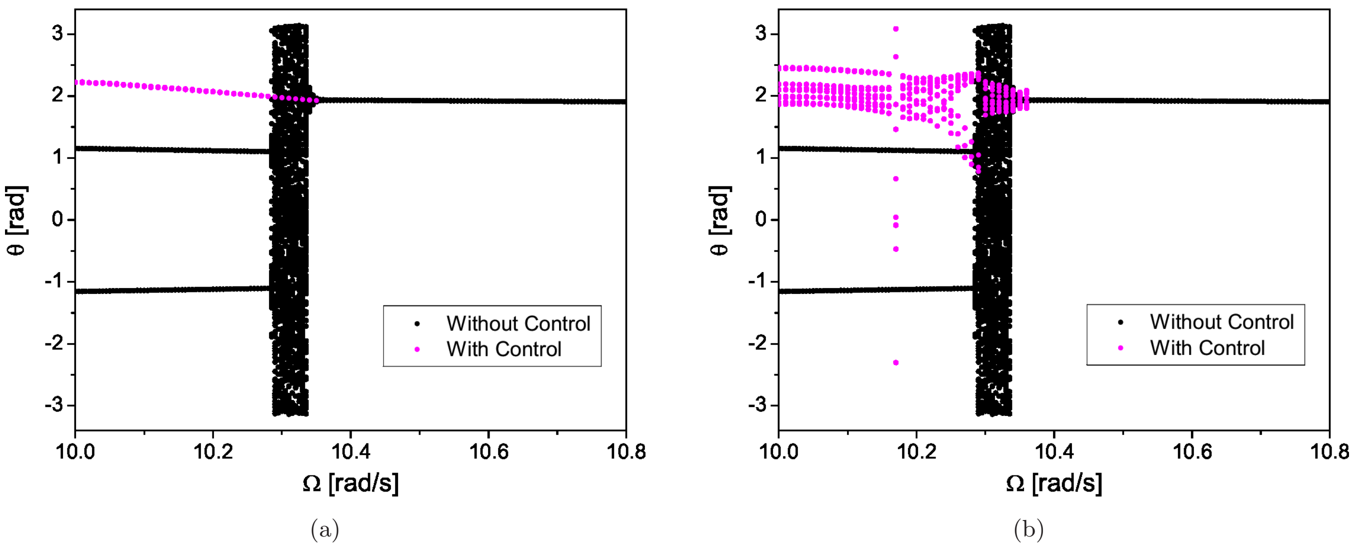


Fig. 12. Bifurcation diagram at $E_0 = 85\,\text{V}$ with and without control: (a) with a wait time, (b) without a wait time.

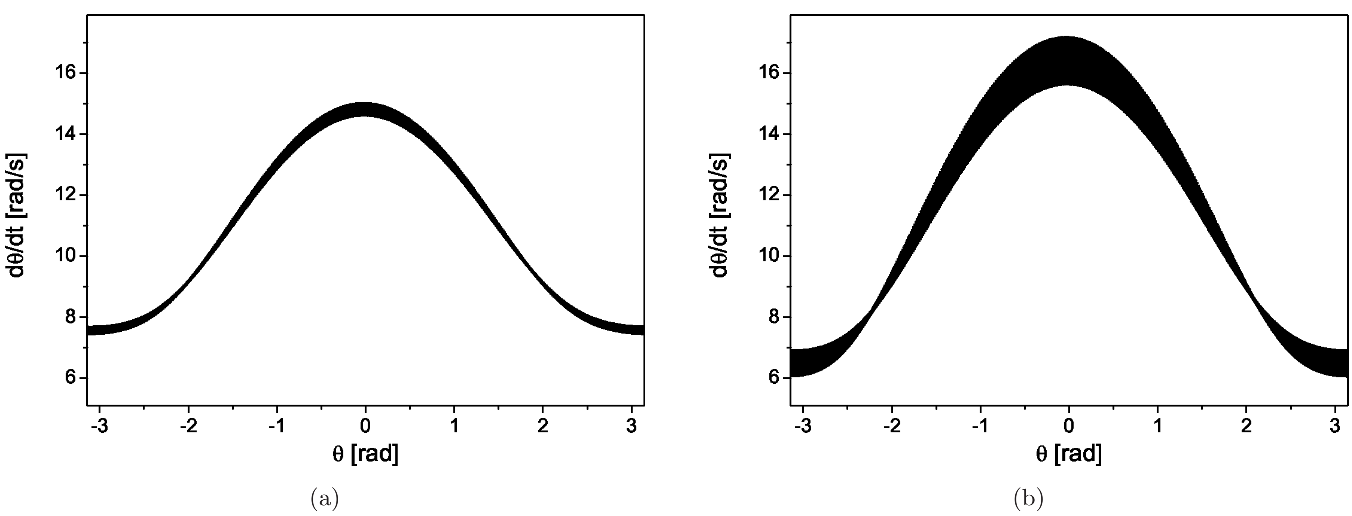


Fig. 13. Steady state response of the controlled system for $E_0 = 85\,\text{V}$. (a) $\Omega = 10.3\,\text{rad/s}$, (b) $\Omega = 10.15\,\text{rad/s}$.

chaotic motion is avoided as well as period-2 orbit. Nevertheless, it should be pointed out that the controlled response is a quasi-periodic behavior.

5. Conclusions

This paper presents the modeling and analysis of bifurcation control of a pendulum-shaker system. This electro-mechanical system may be used to simulate the dynamics of a pendulum system used to explore the concept of energy harvesting from sea waves. This idea requires the rotating periodic response of the system to be maintained throughout a significant parameters range. The bifurcation control is applied in order to stabilize desired rotational orbits for the parameter values where they are originally unstable.

The continuous control method known as extended time delayed feedback is employed and different scenarios are investigated. First, the classical chaos control is realized, where some UPOs embedded in chaotic attractor are stabilized. Then period doubling bifurcation is prevented in order to maintain a period-1 rotating orbit. Finally, bifurcation to chaos is avoided and a stable rotating solution is maintained.

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