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Comparative analysis of chaos control methods: A mechanical system case study

Aline Souza de Paula [a], Marcelo Amorim Savi [b,*]

[a] Universidade de Brasília, Department of Mechanical Engineering, 70.910.900, Brasília, DF, Brazil
[b] Universidade Federal do Rio de Janeiro, COPPE, Department of Mechanical Engineering, 21.941.972, Rio de Janeiro, RJ, P.O. Box 68.503, Brazil

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ABSTRACT

Chaos may be exploited in order to design dynamical systems that may quickly react to some new situation, changing conditions and their response. In this regard, the idea that chaotic behavior may be controlled by small perturbations allows this kind of behavior to be desirable in different applications. This paper presents an overview of chaos control methods classified as follows: OGY methods – include discrete and semi-continuous approaches; multiparameter methods – also include discrete and semi-continuous approaches; and time-delayed feedback methods that are continuous approaches. These methods are employed in order to stabilize some desired UPOs establishing a comparative analysis of all methods. Essentially, a control rule is of concern and each controller needs to follow this rule. Noisy time series is treated establishing a robustness analysis of control methods. The main goal is to present a comparative analysis of the capability of each chaos control method to stabilize a desired UPO.

1. Introduction

Non-linearities are responsible for a great variety of possibilities in natural systems. Chaos is one of these possibilities being related to an intrinsic richness. A geometrical form to understand chaos is related to a transformation known as Smale horseshoe that establishes a sequence of contraction–expansion–folding which causes the existence of an infinity number of unstable periodic orbits (UPOs) embedded in a chaotic attractor. This set of UPOs constitutes the essential structure of chaos. Besides, chaotic behavior has other important aspects as sensitive dependence to initial conditions and ergodicity.

These aspects of chaos may be exploited in order to design dynamical systems that may quickly react to some new situation, changing conditions and their response. Under this condition, a dynamical system adopting chaotic regimes becomes interesting due to the wide range of potential behaviors being related to a flexible design. The idea that chaotic behavior may be controlled by small perturbations applied in some system parameters allows this kind of behavior to be desirable in different applications.

In brief, chaos control methods may be classified as discrete and continuous methods. Semi-continuous method is a class of discrete method that lies between discrete and continuous method. The pioneer work of Ott et al. [27] introduced the basic idea of chaos control proposing the discrete OGY method. Afterwards, Hübinger et al. [20] proposed a variation of the OGY technique considering semi-continuous actuations in order to improve the original method capacity to stabilize unstable orbits. Pyragas [29] proposed a continuous method that stabilizes UPOs by a feedback perturbation proportional to the difference between the present and a delayed state of the system.

This article deals with a comparative analysis of chaos control methods that are classified as follows: OGY methods – include discrete and semi-continuous approaches [27,20]; multiparameter methods – also include discrete and semi-continuous approaches [10,11]; and time-delayed feedback methods that are continuous approaches [29,34]. Fig. 1 presents an overview of chaos control methods analyzed in this work.

Many research efforts were presented in literature in order to improve the originals chaos control techniques and there are numerous review papers concerning these procedures. In this regard, Shinbrot et al. [33], Ditto et al. [14], Grebogi and Lai [18] and Dubé and Després [15] discussed concepts of chaos and its control presenting discrete chaos control techniques based on OGY method. Pyragas [30] presented an overview of continuous chaos control methods based on time-delayed feedback and mentioned several numerical and experimental applications. Ogorzalek [25], Arecchi et al. [3] and Fradkov and Evans [16] presented review articles that furnish a general overview of chaos control methods, including discrete and continuous techniques. Besides these methods, Boccaletti et al. [6] also treated tracking and synchronization

* Corresponding author.
E-mail addresses: alinedepaula@unb.br (A.S. de Paula), savi@mecanica.ufrj.br (M.A. Savi).

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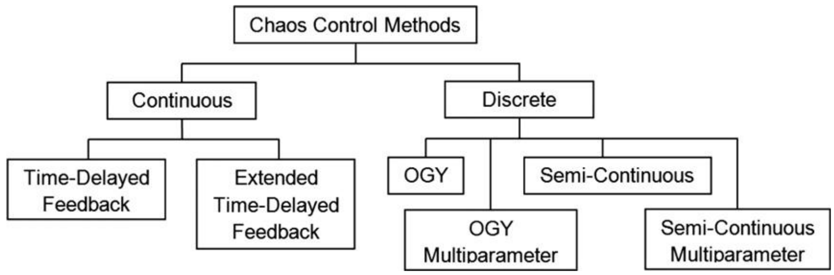

Fig. 1. Chaos control methods.

of chaotic systems and mentioned several experimental implementations. Andrievskii and Fradkov [1] discussed several methods for controlling chaotic systems including chaos control techniques and traditional control methods, while Andrievskii and Fradkov [2] mentioned several works that apply these control procedures to numerous systems of different fields. Fradkov et al. [17] and Savi et al. [32] presented reviews focused on chaos control methods applied to mechanical systems.

Recently, different approaches are being employed in order to stabilize chaotic behavior. In this regard, Kapitaniak [22] applied non-feedback methods by adding a controller, which consists in a linear oscillator, to the dynamical system with the help of coupling elements. Chen [7] presented the design of linear and non-linear conventional feedback controllers based on Lyapunov function methods in other to stabilize chaotic behavior. Bessa et al. [5] proposed an adaptive fuzzy sliding mode strategy enhanced by an adaptive fuzzy algorithm to cope with modeling inaccuracies. The method is applied in order to stabilize UPOs embedded in chaotic response as well as generic orbits.

Despite the numerous review papers concerning the control of chaos, there is a lack of reports that present a comparative analysis of the control strategies, which is the main goal of this contribution. The capability of the chaos control methods to stabilize a desired UPO is analyzed in this paper. A mechanical system is of concern as an application of the general procedure and all signals are generated by numerical integration of a mathematical model, using experimentally identified parameters. In order to consider a system with high instability, a non-linear pendulum treated in other references is considered [11,12,28]. Noise influence is treated by considering signals with observation noise. Results show the performance of each method to stabilize desired orbits exploring some limitations and its application.

The paper is organized as follows. Initially, a brief introduction of chaos control methods is presented. Afterwards, a comparative study is carried out by defining some control rules that should be followed by each controller. Noise influence is treated in the sequence showing the robustness of each controller. Finally, the paper presents the concluding remarks.

2. Chaos control methods

The control of chaos can be treated as a two-stage process. The first stage is called learning stage where the UPOs are identified and system parameters necessary for control purposes are chosen. A good alternative for the UPO identification is the close return method [4]. This identification is not related to the knowledge of the system dynamics details being possible to use time series analysis. The estimation of system parameters is done in different ways for discrete and continuous methods. After the learning stage, the second stage starts promoting the UPO stabilization employing chaos control methods that are discussed in this section.

2.1. OGY method

The OGY method [27] is described by considering a discrete system of the form of a map $\xi^{n+1}=F(\xi^n,p^n)$, where $p\in\Re$ is an accessible parameter for control. This is equivalent to a parameter dependent map associated with a general surface, usually a Poincaré section. Let $\xi_C^{n+1}=F(\xi_C^n,p_0)$ denotes the unstable fixed point on this section corresponding to an unstable periodic orbit in the chaotic attractor that one wants to stabilize. Basically, the control idea is to monitor the system dynamics until the neighborhood of this point is reached. When this happens, a proper small change in the parameter p causes the next state ξ^{n+1} to fall into the stable direction of the fixed point. In order to find the proper variation in the control parameter, δp, it is considered a linearized version of the dynamical system in the neighborhood of the equilibrium point given by Eq. (1). The linearization has a homeomorphism with the non-linear problem that is assured by the Hartman–Grobman theorem [19,36,21,35,31]:

$$\Delta\xi^{n+1}=J^n\Delta\xi^n+w^n\Delta p^n \tag{1}$$

where $\Delta\xi^n=\xi^n-\xi_C^n$, $\Delta\xi^{n+1}=\xi^{n+1}-\xi_C^{n+1}$, and $\Delta p^n=p^n-p_0$. $J^n=D_{\xi^n}\ F(\xi^n,P^n)|_{\xi^n=\xi_C^n,P^n=P_0}$ is the Jacobian matrix and $w^n=D_{p^n}F(\xi^n,\ p^n)|_{\xi^n=\xi_C^n,p^n=p_0}$ is the sensitivity vector.

Fig. 2 presents a schematic picture that allows a geometrical comprehension of the stabilization process. Since the chaotic behavior is related to a saddle point, it is possible to visualize this stabilization over a saddle.

Hübinger et al. [20] verified that the linear mapping J^n deforms a sphere around ξ_C^n into an ellipsoid around ξ_C^{n+1}. Therefore, a singular value decomposition (SVD) can be employed in order to determine the unstable and stable directions, v_u^n and v_s^n, in Σ_n which are mapped onto the largest, $\sigma_u^n u_u^n$, and shortest, $\sigma_s^n u_s^n$, semi-axis of the ellipsoid in Σ_{n+1}, respectively. Here, σ_u^n and σ_s^n are the singular values of J^n:

$$J^n=U^nW^n(V^n)^T=\left\{u_u^n\quad u_s^n\right\}\begin{bmatrix}\sigma_u^n & 0\\ 0 & \sigma_s^n\end{bmatrix}\left\{v_u^n\quad v_s^n\right\}^T \tag{2}$$

Korte et al. [23] established the control target as being the adjustment of δp^n such that the direction v_s^{n+1} on the map $n+1$ is obtained, resulting in a maximal shrinking on map $n+2$. Therefore, it demands $\Delta\xi^{n+1}=\alpha v_s^{n+1}$, where $\alpha\in\Re$. Hence

$$J^n\Delta\xi^n+w^n\Delta p^n=\alpha v_s^{n+1} \tag{3}$$

from which α and δp^n can be conveniently chosen.

The OGY method can be employed even in situations where a mathematical model is not available. Under this situation, all parameters can be extracted from time series analysis. The

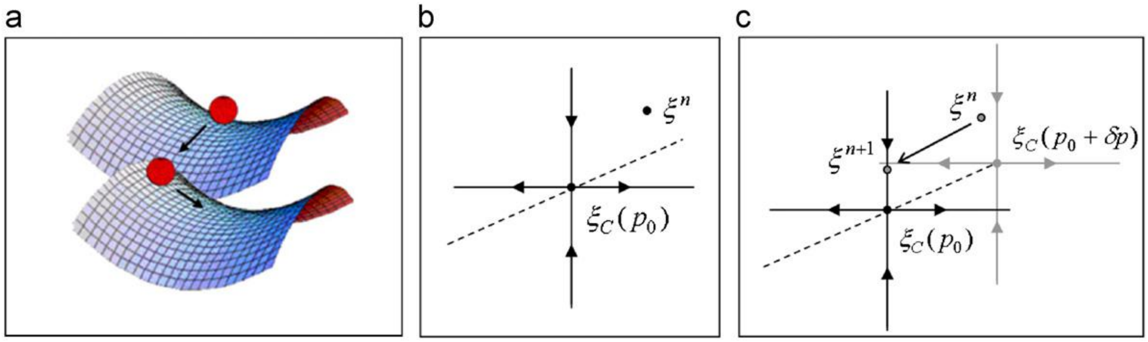

Fig. 2. OGY method: (a) schematic picture of the method; (b) ξ^n in the control neighborhood; and (c) ξ^{n+1}over the stable directions due to perturbation δp.

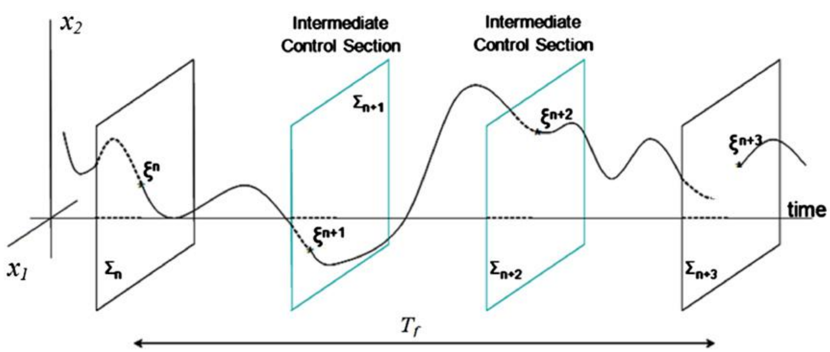

Fig. 3. Semi-continuous method.

Jacobian J^n and the sensitivity vector w^n can be estimated from a time series using a least-square fit method as described in Auerbach et al. [4] and Otani and Jones [26].

Otani and Jones [26] presented some important aspects of the OGY method. As positive points, they mentioned the use of small perturbations for stabilization, the flexibility due to chaos, independence from equations of motion, high computational efficiency, and robustness due to parameter uncertainties. As drawbacks, the authors mentioned the difficulty to stabilize either orbits with high periodicity or systems with high instability, and the necessity to wait the system to visit the neighborhood of some UPO. An alternative to deal with some of the OGY drawbacks is the use of as many control stations as it is necessary to stabilize some orbits. This is the essential point related to semi-continuous method.

2.1.1. *Semi-continuous method*

The semi-continuous method (SC) lies between the continuous and the discrete time control because one can introduce as many intermediate Poincaré sections, viewed as control stations, as it is necessary to achieve stabilization of a desired UPO [20]. Nevertheless, the response time needs to be considered and usually limits the distance between two control stations [28]. Therefore, the SC method is based on measuring transition maps of the system. These maps relate the state of the system in one Poincaré section to the next.

In order to use N control stations per forcing period T, one introduces N equally spaced successive Poincaré sections Σ_n, $n=0, ..., (N-1)$. Let $\xi_C^n \in \Sigma_n$ be the intersections of the UPO with Σ_n and F be the mapping from one control station Σ_n to the next one Σ_{n+1} (Fig. 3).

2.2. *Multiparameter method*

The multiparameter chaos control method (MP) is based on the OGY approach and considers N_p different control parameters, p_i $(i=1, ..., N_p)$. Moreover, only one of these control parameters actuates in each control station [10,11]. Under this assumption, the map F that establishes the relation of the system behavior between the control stations Σ_n and Σ_{n+1}, depends on all control parameters. Although only one parameter actuates in each section, it is considered the influence of all control parameters based on their positions in station Σ_n. On this basis

$$\xi^{n+1}=F(\xi^n,P^n) \tag{4}$$

where P^n is a vector with all control parameters. By using a first order Taylor expansion, one obtains the linear behavior of the map F in the neighborhood of the control point ξ_C^n and around the control parameter reference position, P_0, which is defined by

$$\Delta\xi^{n+1}=J^n\Delta\xi^n+W^n\Delta P^n \tag{5}$$

where $\Delta P^n=P^n-P_0$ is related to the control actuation. It is important to mention that in the sensitivity matrix, W^n, each column is related to a specific control parameter. In order to evaluate the influence of all parameters actuation, it is assumed that the system response to all parameters perturbation is given by a linear combination of the system responses when each parameter actuates isolated and the others are fixed at their reference value. Therefore

$$\Delta P^n=B^n\Delta p^n \tag{6}$$

where B^n is defined as a $[N_p\times N_p]$ diagonal matrix formed by the weighting parameters, i.e., $diag(B^n)_i=\beta_i^n$. This can be understood by considering that each parameter influence is related to a vector with components $q_i=W_i^n\Delta p_i^n=W_i^n(p_i^n-p_{0i})$, and the general perturbation is given by

$$q=\beta_1 q_1+\beta_2 q_2+\cdots+\beta_{N_p} q_{N_p}=W^n B^n\Delta p^n \tag{7}$$

Moreover, by assuming that only one parameter actuates in each control station it is possible to define active parameters, represented by subscript a, $\Delta P_a^n = B_a^n \Delta p_a^n$ (actuate in station Σ_n), and passive parameters, represented by subscript p, $\Delta P_p^n = B_p^n \Delta p_p^n$ (do not actuate in station Σ_n). At this point, it is assumed a weighting matrix for active parameter, B_a^n, and other for passive parameters, B_p^n. Therefore

$$\Delta\xi^{n+1} = J^n \Delta\xi^n + W^n \Delta P_a^n + W^n \Delta P_p^n \tag{8}$$

Now, it is necessary to align the vector $\delta\xi^{n+1}$ with the stable direction v_s^{n+1} ($\Delta\xi^{n+1} = \alpha v_s^{n+1}$) where $\alpha \in \Re$, that needs to satisfy:

$$J^n \Delta\xi^n + W^n \Delta P_a^n + W^n \Delta P_p^n = \alpha v_s^{n+1} \tag{9}$$

Therefore, once the unknown variables are α and the non-vanishing term of the vector ΔP_a^n, one obtains the following system:

$$\begin{bmatrix} \Delta P_{ai}^n \\ \alpha \end{bmatrix} = -\begin{bmatrix} W_i^n & -v_s^{n+1} \end{bmatrix}^{-1} \begin{bmatrix} J^n & W^n \end{bmatrix} \begin{bmatrix} \Delta\xi^n \\ \Delta P_p^n \end{bmatrix} \tag{10}$$

where ΔP_{ai}^n is related to the non-vanishing element of the vector ΔP_a^n that consists in the active parameter in Σ_n, and W_i^n corresponds to the sensitivity matrix column related to this active parameter. The solution of this system furnishes the necessary values for the system stabilization and it is important to note that the real perturbation is given by $\Delta p_{ai}^n = \Delta P_{ai}^n / \beta_{ai}^n$.

A particular case of this control procedure has uncoupled control parameters meaning that each parameter returns to the reference value when it becomes passive. Moreover, since there is only one active parameter in each control station, the system response to parameter perturbation is the same as when it actuates alone. Under this assumption, passive influence vanishes and active vector is weighted by 1, which is represented by

$$B_p^n = 0 \text{ and } B_a^n = I \tag{11}$$

where I is the identity matrix.

Therefore, the map F is just a function of the active parameters, $\xi^{n+1} = F(\xi^n, P_a^n)$, and the linear behavior of the map F in the neighborhood of the control point ξ_C^n and around the control parameter reference positions, P_0, is now defined by

$$\Delta\xi^{n+1} = J^n \Delta\xi^n + W^n \Delta P_a^n \tag{12}$$

where the sensitivity matrix W^n is the same of the previous case. Moreover, since $B_a^n = I$, it follows that $\Delta P_a^n = \Delta p_a^n$, thus the value of ΔP_a^n corresponds to the real perturbation necessary to stabilize the system.

The difference between the multiparameter method (MP) [10] and the semi-continuous multiparameter method (SC-MP) [11] is that the first considers only one control station per forcing period while the other considers as many control stations as necessary to stabilize the system per forcing period. Therefore, the SC-MP is the general case that can represent the MP when only one control station per period is of concern. In the same way, the OGY can be seen as a particular case when only one control station and only one control parameter are considered.

2.3. Time-delayed feedback methods

Continuous methods for chaos control were first proposed by Pyragas [29] and are based on continuous-time perturbations to perform chaos control. This control technique deals with a dynamical system modeled by a set of ordinary non-linear differential equations as follows:

$$\dot{x} = Q(x,t) + B(t) \tag{13}$$

where $x = x(t) \in R^n$ is the state variable vector, $Q(x,t) \in R^n$ defines the system dynamics, while $B(t) \in R^n$ is associated with the control action.

Socolar et al. [34] proposed a control law named as the extended time-delayed feedback control (ETDF) considering the information of time-delayed states of the system represented by

$$B(t) = K[(1-R)S_\tau - x], \quad S_\tau = \sum_{m=1}^{N_\tau} R^{m-1} x_{m\tau} \tag{14}$$

where $K \in R^{n \times n}$ is the feedback gain matrix, $0 \le R < 1$, $S_\tau = S(t-\tau)$ and $x_{m\tau} = x(t - m\tau)$, τ is the time delay. The UPO stabilization can be achieved by a proper choice of R and K. Note that for any R and K, perturbation of Eq. (14) vanishes when the system trajectory is on any UPO since $x(t-m\tau) = x(t)$ for all m if $\tau = T_i$, where T_i is the periodicity of the ith UPO. It should be pointed out that when $R=0$, the ETDF turns into the original time-delayed feedback control method (TDF) proposed by Pyragas [29] where the control law is based on a feedback of the difference between the current and a delayed state given by

$$B(t) = K[x_\tau - x] \tag{15}$$

The controlled dynamical system consists of a set of delay differential equations (DDEs). The solution of this system can be done by different procedures. An interesting alternative is to establish an initial function $x_0 = x_0(t)$ over the interval $(-N_\tau\tau, 0)$. This function can be estimated by a Taylor series expansion as proposed by Cunningham [8]:

$$x_{m\tau} = x - m\tau\,\dot{x} \tag{16}$$

Under this assumption, the following system is obtained:

$$\dot{x} = Q(x,t) + K[(1-R)S_\tau - x], \quad \text{where} \begin{cases} S_\tau = \sum_{m=1}^{N_\tau} R^{m-1}[x - m\tau\dot{x}], & \text{for } (t - N_\tau\tau) < 0 \\ S_\tau = \sum_{m=1}^{N_\tau} R^{m-1} x_{m\tau}, & \text{for } (t - N_\tau\tau) \ge 0 \end{cases} \tag{17}$$

Note that DDEs contain derivatives that depend on the solution at delayed time instants. Therefore, besides the special treatment that must be given for $(t - N_\tau\tau) < 0$, it is necessary to deal with time-delayed states while integrating the system. A fourth-order Runge–Kutta method with linear interpolation on the delayed variables is employed in this work for the numerical integration of the controlled dynamical system [24].

An important difference between continuous and discrete methods is that in continuous methods it is not necessary to wait the system to visit the neighborhood of the desired orbit. Another particular characteristic related to the learning stage is that, besides the UPO identification common to all control methods, it is necessary to establish proper values of the controller parameters, R and K, for each desired orbit. This choice is done by analyzing Lyapunov exponents of the UPO, establishing negative values of the largest Lyapunov exponent. After this first stage, the control stage is performed, where the desired UPOs are stabilized. De Paula and Savi [12] discussed a proper procedure to evaluate the largest Lyapunov exponents necessary for the controller parameters.

3. Comparative analysis

As an application of the general chaos control methods, a system with high instability characteristic is of concern. A non-linear pendulum actuated by two different control parameters is considered, as presented in schematic pictures of Fig. 4. The motivation of the proposed pendulum is an experimental set up discussed in De Paula et al. [9] that proposed a mathematical model to describe the pendulum dynamical behavior. Basically,

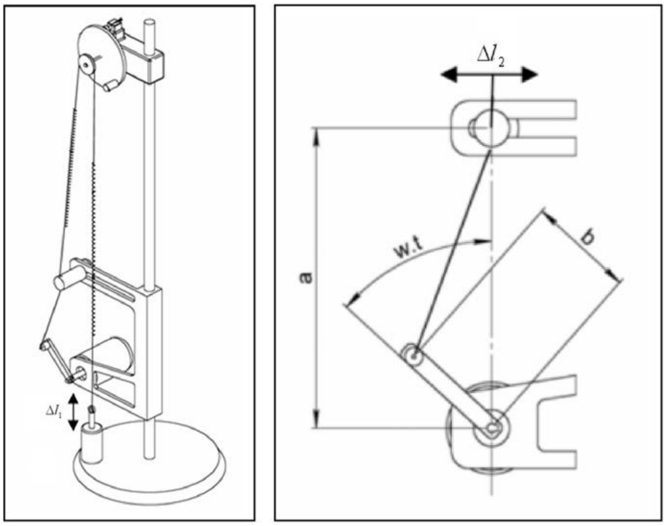

Fig. 4. Non-linear pendulum schematic pictures.

the pendulum consists of an aluminum disc with a lumped mass. An electric motor harmonically excites the pendulum via a string-spring device, which provides torsional stiffness to the system. Fig. 4 also presents the actuators responsible to promote system perturbations (Δl_1 and Δl_2).

The mathematical model for the pendulum dynamics describes the time evolution of the angular position, ϕ, assuming that ϖ is the forcing frequency, I is the total inertia of rotating parts, k is the spring stiffness, ζ represents the viscous damping coefficient, μ the dry friction coefficient, m is the lumped mass, a defines the position of the guide of the string with respect to the motor, b is the length of the excitation arm of the motor, D is the diameter of the metallic disc, and d is the diameter of the driving pulley. The equation of motion is given by [9]

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{kd^2}{2I} & -\frac{\zeta}{I} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ \frac{kd}{2I}(\Delta f(t) - \Delta l_1) - \frac{mgDsen(x_1)}{2I} - \frac{2\mu}{\pi I}\arctan(qx_2) \end{bmatrix} \quad (18)$$

where $\Delta f(t) = \sqrt{a^2+b^2+\Delta l_2{}^2 - 2ab\cos(\varpi t) - 2b\Delta l_2 \sin(\varpi t)} - (a-b)$ and Δl_1 and Δl_2 correspond to actuations.

Numerical simulations of the pendulum dynamics are in close agreement with experimental data by assuming parameters used in De Paula et al. [9]: $a = 1.6 \times 10^{-1}$ m; $b = 6.0 \times 10^{-2}$ m; $d = 4.8 \times 10^{-2}$ m; $D = 9.5 \times 10^{-2}$ m; $m = 1.47 \times 10^{-2}$ kg; $I = 1.738 \times 10^{-4}$ kg m^2; $k = 2.47$ N/m; $\zeta = 2.368 \times 10^{-5} kg m^2 s^{-1}$; $\mu = 1.272 \times 10^{-4}$N m; and $\omega = 5.61$ rad/s. This set of parameters is related to chaotic behavior. Numerical simulations are carried out in order to generate position and velocity time series used in chaos control analysis. UPOs embedded in chaotic attractor are identified by using the close return method [4]. This identification consists in the first step of the learning stage being common to all control methods.

This section establishes a comparative analysis of chaos control methods that, in principle, are capable to perform UPO stabilization of the non-linear pendulum. Due to system instability, the OGY method is not capable to perform the system stabilization even though an orbit with low periodicity is of concern. The MP coupled approach presents a better performance in contrast with the single-parameter approach being able to stabilize a period-1 UPO. The MP uncoupled approach, however, is not capable to present the stabilization as well. De Paula and Savi [10] showed some situations where the MP method presents better performance than other methods.

In this regard, the comparative analysis deals with only four different controllers: semi-continuous (SC), semi-continuous multiparameter (SC-MP) (coupled and uncoupled approaches), and time-delayed feedback methods (TDF and ETDF). The strategy of analysis considers a control rule that is followed by each controller. Noise influence is also of concern by treating noisy signals.

Before starting the comparative analysis, let us highlight some important aspects. Proper values for parameters β_a and β_p in coupled MP method are defined by the brute-force approach, as described in De Paula and Savi [11], which states that $\beta_p = 2.5$ and $\beta_p = 1.5$. On the other hand, the uncoupled approach avoids this kind of evaluation since $\beta_a = 1$ and $\beta_p = 0$. Concerning continuous methods, only the first control parameter, Δl_1, is used to promote perturbations in the system. Moreover, it is considered only the dependence of delayed states of x_2 in the control law. Under these assumptions, gain K is a scalar.

3.1. Control methods performance

Comparative analysis evaluates the performance of the SC, the SC-MP, coupled and uncoupled approaches, and the ETDF comparing the efficacy of each one to stabilize UPOs. With this aim, a control rule is defined for the stabilization of four different UPO in the following sequence: a period-5 orbit during the first 500 periods, a period-3 from period 500 to 1000, a period-8 from 1000 to 1500, and finally a period-1, from period 1500 to 2000. Fig. 5 presents these four UPOs in one of the control sections considered by the semi-continuous methods, while Fig. 6 shows the UPOs in phase space.

Initially, the SC is employed by considering the isolated perturbation performed by the parameters Δl_1 and Δl_2. Four control stations per forcing period are considered by assuming maximum perturbation of $|\Delta l_{1\,max}| = 15$ mm and $|\Delta l_{2\,max}| = 25$ mm with reference position of $\Delta l_{10} = \Delta l_{20} = 0$ mm. Fig. 7(a) shows the desired trajectory, imposed by the control rule, and the system time evolution at control station #1 controlled by parameter Δl_1, while Fig. 7(b) shows the actuator perturbations in the same control station. On the other hand, Fig. 8 presents the same pictures by assuming the perturbation of parameter Δl_2. It should be noticed that both procedures are not capable to follow all control rule (three of the four UPOs are stabilized). Moreover, before the stabilization of UPO is achieved it can be observed a region related to chaotic

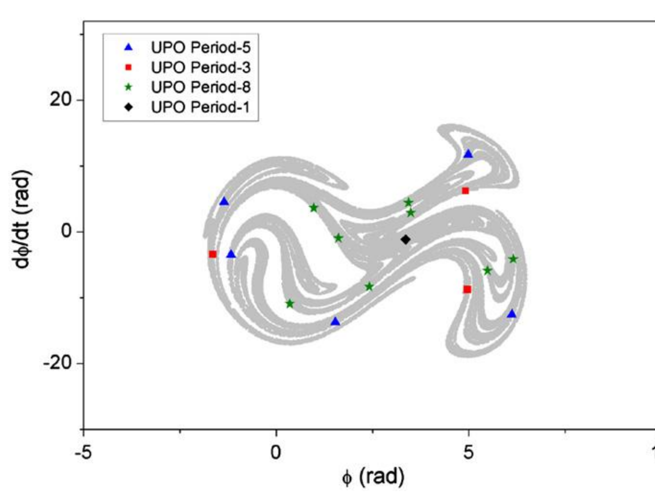

Fig. 5. UPOs of the control rule at control station #1.

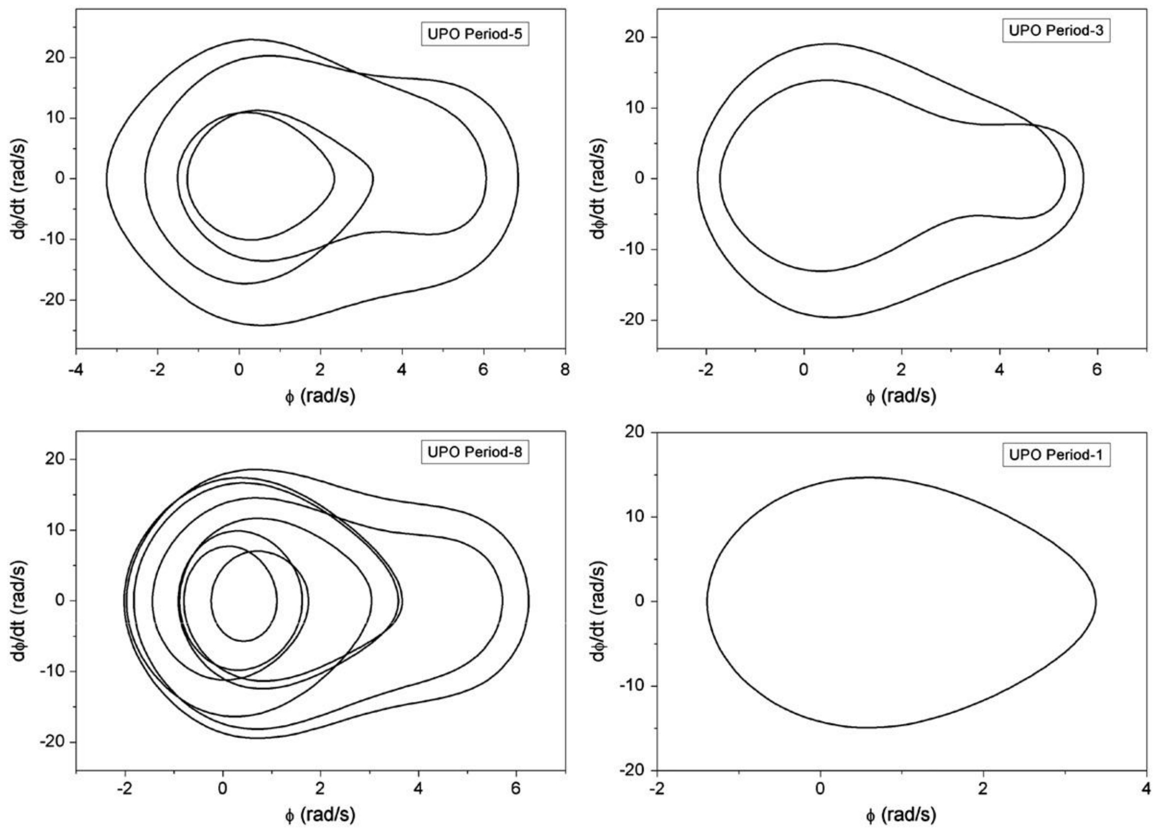

Fig. 6. UPOs of the control rule.

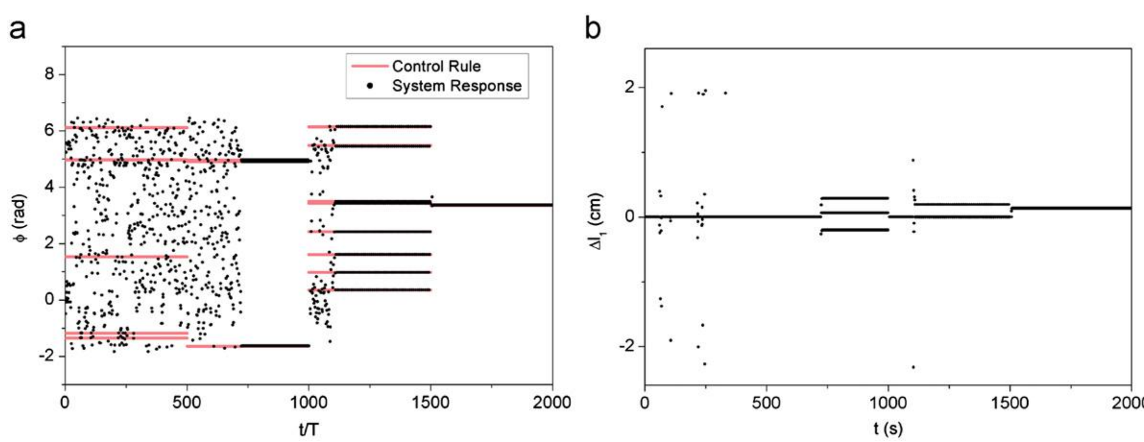

Fig. 7. System controlled using SC with parameter Δl_1 at the control station #1: (a) system displacement and desired trajectory and (b) perturbation.

behavior that corresponds to the wait time that system dynamics takes to reach the neighborhood of desired control point.

At this point, the coupled approach of the SC-MP is employed in order to stabilize the non-linear pendulum UPOs using two control parameters. Once again, four control stations per forcing period are considered and maximum perturbation of $|\Delta l_{1\ max}| = 5\,\text{mm}$ and $|\Delta l_{2\ max}| = 15\,\text{mm}$ are assumed with reference position being $\Delta l_{10}{=}\Delta l_{20}{=}0$ mm. Figs. 9(a) and 10(a) show the desired trajectory, imposed by the control rule, and the system time evolution at control stations #1 and #2, respectively, while Figs. 9(b) and 10(b) show the actuators behavior in the same control stations. These results show that this control approach is effective to stabilize all orbits of the control rule.

The uncoupled approach of the SC-MP is now focused on in order to follow the control rule using two control parameters. Four control stations per forcing period are considered using the same maximum perturbation and reference position of the coupled approach. Figs. 11(a) and 12(a) show the desired trajectories, imposed by the control rule, and the system time evolution at control stations #1 and #2, respectively, while Figs. 11(b) and 12(b) show the actuators behavior in the same control stations. As the coupled approach, the uncoupled approach is effective to stabilize all orbits of the control rule.

The extended time-delayed feedback method (ETDF) is now employed to follow the control rule considering the use of parameter Δl_1 with maximum perturbation of $|\Delta l_{1\ max}| = 5\,\text{mm}$ with reference position being $\Delta l_{10}{=}0$ mm. Since the ETDF is a

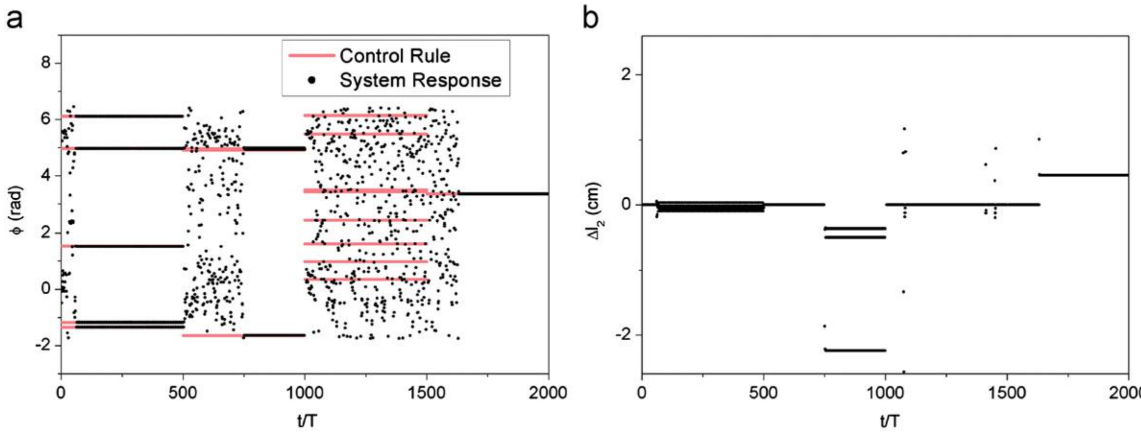

Fig. 8. System controlled using SC with parameter Δl_2 at the control station #1: (a) system displacement and desired trajectory and (b) perturbation.

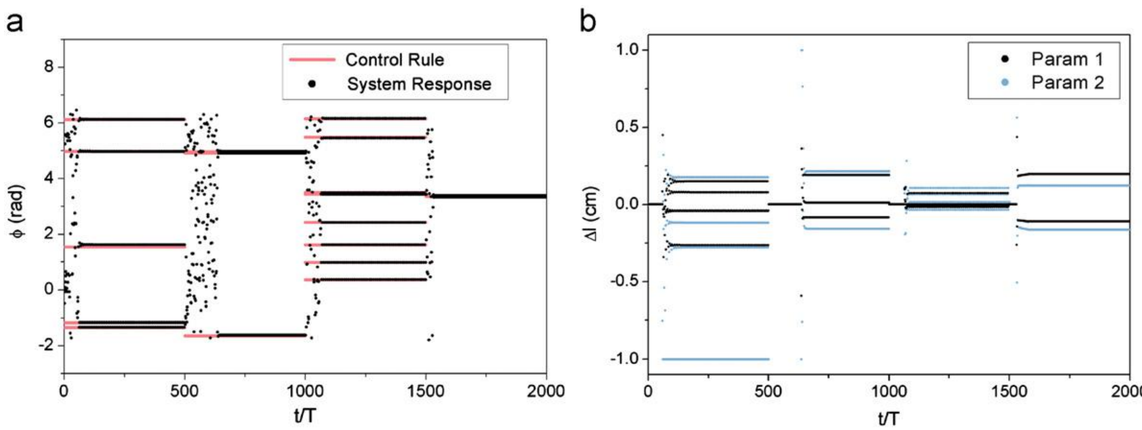

Fig. 9. SC-MP coupled approach evaluation: (a) system response, desired trajectory and (b) control action at control station #1.

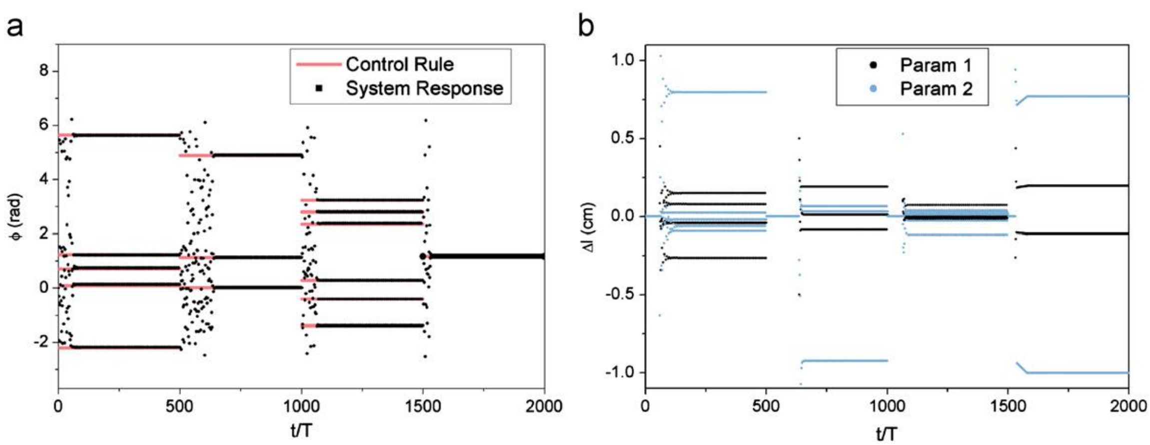

Fig. 10. SC-MP coupled approach evaluation: (a) system response, desired trajectory and (b) control action at control station #2.

continuous method, it is not necessary to consider control stations because perturbations are applied to the system at each time step. The stabilization results, however, are presented in control stations in order to establish a comparison with results obtained from the semi-continuous methods. Fig. 13(a) shows the desired trajectory, imposed by the control rule, and the system time evolution at control station #1, while Fig. 13(b) shows the actuator behavior in the same control station. Note that the ETDF is not able to stabilize the first and the third orbits of the control rule. Besides, the second orbit of the control rule that is stabilized is different from the identified UPO, as presented in Fig. 14. It is important to highlight that in this work no wait time is considered to start control action, different from SC methods where the wait time is essential. However, De Paula et al. [13] states that the performance of ETDF method can be improved by waiting the system trajectory to fall in the neighborhood of the desired UPO.

The stabilization of a different UPO can be explained by analyzing the values of the maximum Lyapunov exponent.

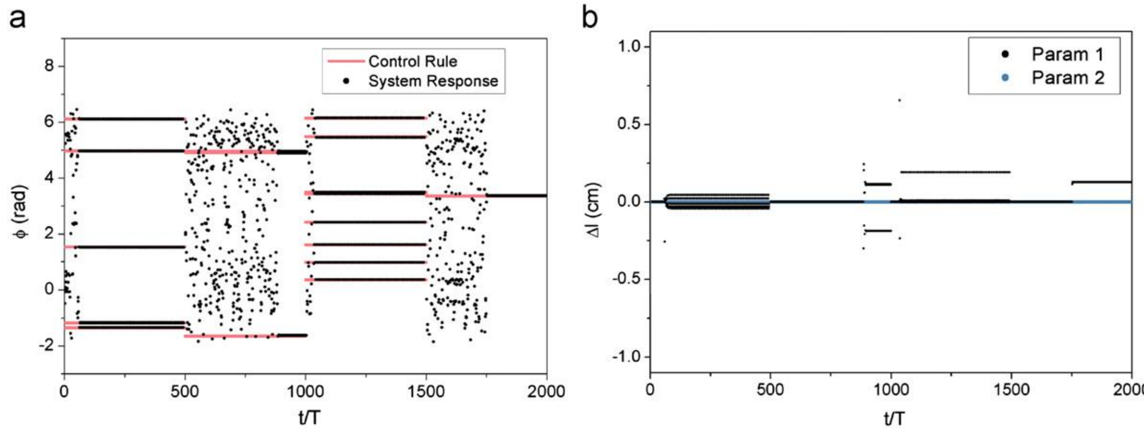

Fig. 11. System controlled using SC-MP uncoupled approach at the control station #1: (a) system displacement and desired trajectory and (b) control action.

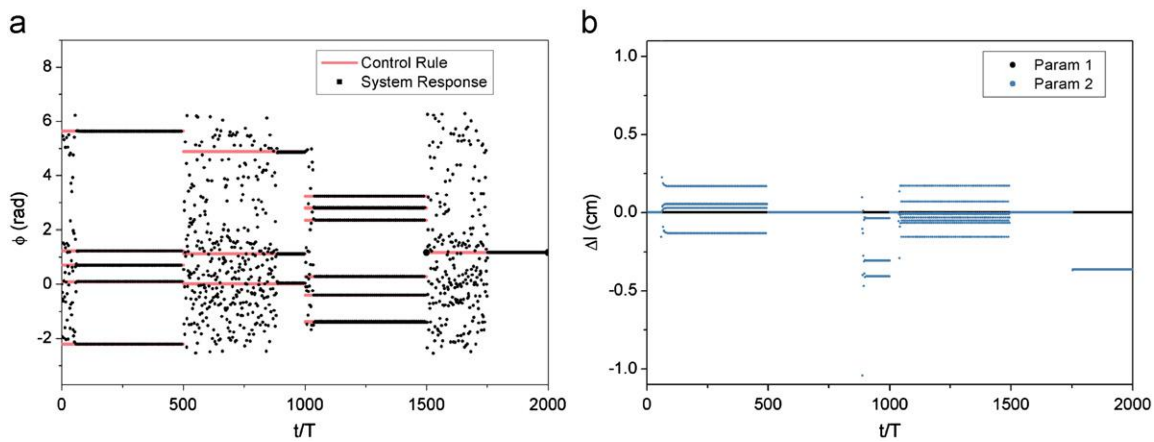

Fig. 12. System controlled using SC-MP uncoupled approach at the control station #2: (a) system displacement and desired trajectory and (b) control action.

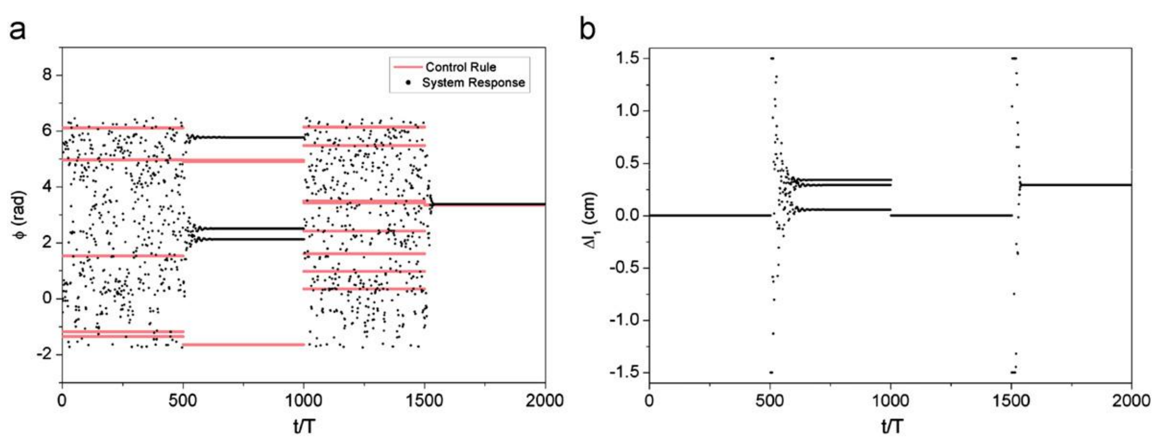

Fig. 13. ETDF evaluation: (a) system displacement, desired trajectory and (b) perturbation at the control station #1.

Although the period-3 UPO of the control rule presents a region with negative Lyapunov exponent for some values of the controller parameters, this region is small and with greater values when compared to the correspondent situation of the stabilized orbit, as shown in Fig. 15. Concerning the first and the third UPOs of the control rule, there are no values of the controller parameters that lead to negative Lyapunov exponent. Therefore, it is not possible to stabilize these orbits by employing the ETDF.

Another possibility concerning the stabilization with continuous methods is that besides the desired orbit, other orbits with submultiple periodicity can be stabilized. This kind of response establishes a difficulty to stabilize a target UPO of high periodicity even employing a proper procedure to evaluate controller parameters [12]. Moreover, some authors point that the increase of K values can stabilize the system. Actually, the use of high values of K can suppress chaos but the stabilized orbit is not necessarily a natural

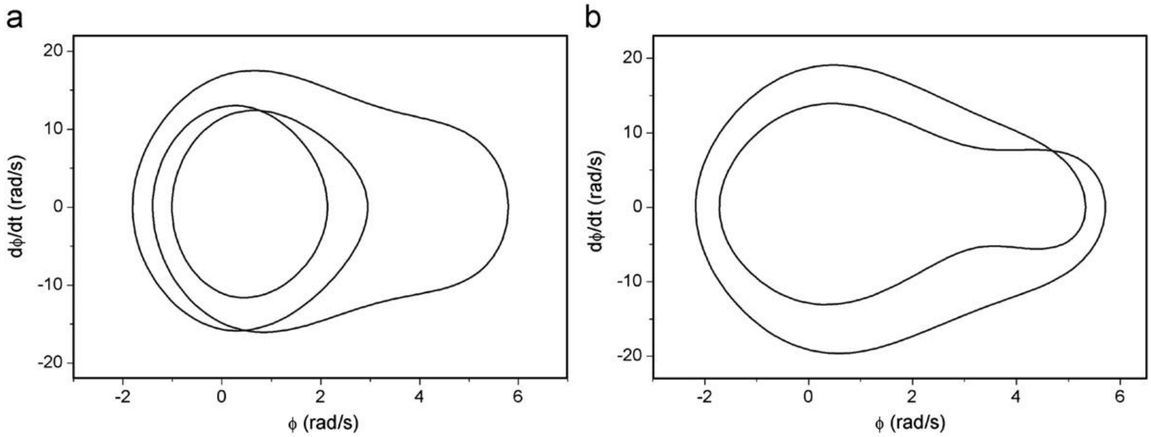

Fig. 14. Period-3 UPO: (a) stabilized from ETDF and (b) desired orbit of the control rule.

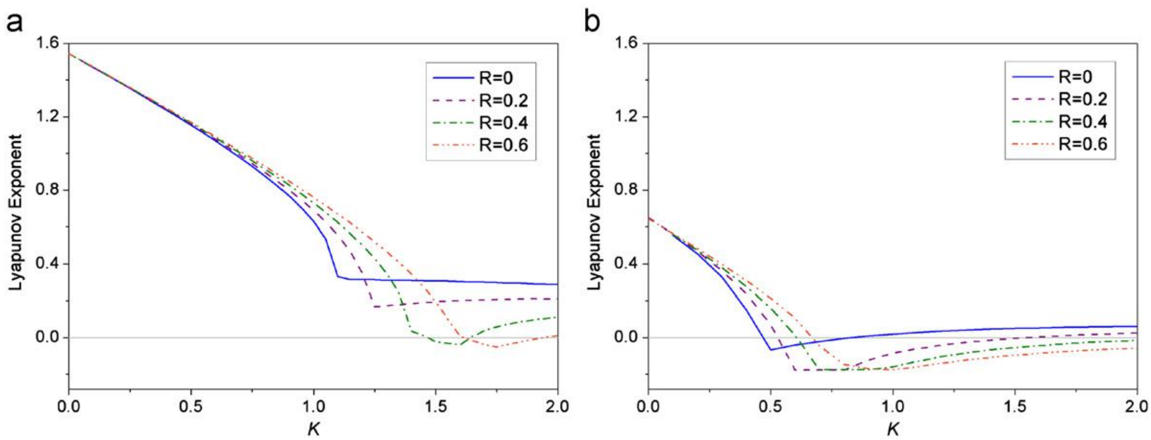

Fig. 15. Maximum Lyapunov exponent: (a) Period-3 UPO of the control rule and (b) stabilized period-3 UPO.

unstable periodic orbit that belongs to chaotic attractor which means that the controller presents higher energy consumption.

3.2. Chaos control performance considering noisy signals

Since noise contamination is unavoidable in experimental data acquisition, it is important to evaluate its effect on chaos control procedures. This section evaluates noise sensitivity of the chaos control techniques previously considered in the comparative analysis: SC, SC-MP, coupled and uncoupled approaches, and ETDF. In order to simulate noisy data sets, a white Gaussian noise is introduced in the signal, comparing results of control procedures with an ideal time series, free of noise. In general, noise can be expressed as follows:

$$\begin{cases} \dot{x} = Q(x,t) + \mu_d, \\ \dot{y} = P(x,t) + \mu_o \end{cases} \quad (19)$$

where x represents state variables, y represents the observed response and $Q(x,t)$ and $P(x,t)$ are non-linear functions. μ_d and μ_o are, respectively, dynamical and observed noises. Notice that μ_d has influence on system dynamics in contrast with μ_o. In this work, it is considered only an observed noise, simulating noise in experimental data due to instrumentation apparatus and, therefore, noise does not have influence in system dynamics.

The noise level can be expressed by the standard deviation, σ, of the system probability Gaussian distribution, that is parameterized by the standard deviation of the clean signal, σ_{signal}, as follows:

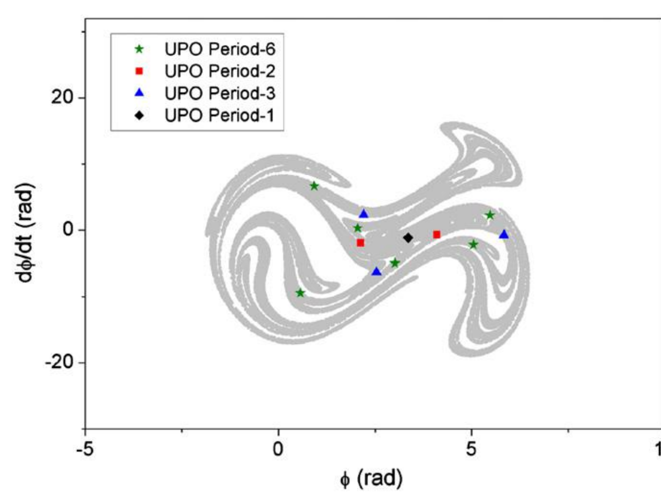

Fig. 16. UPOs of the second control rule at control station #1.

$$\eta\,(\%) = \frac{\sigma}{\sigma_{\text{signal}}} \times 100 \quad (20)$$

A different control rule is assumed in order to compare the control methods performance considering noisy signals. This control rule is defined in order to choose orbits that can be

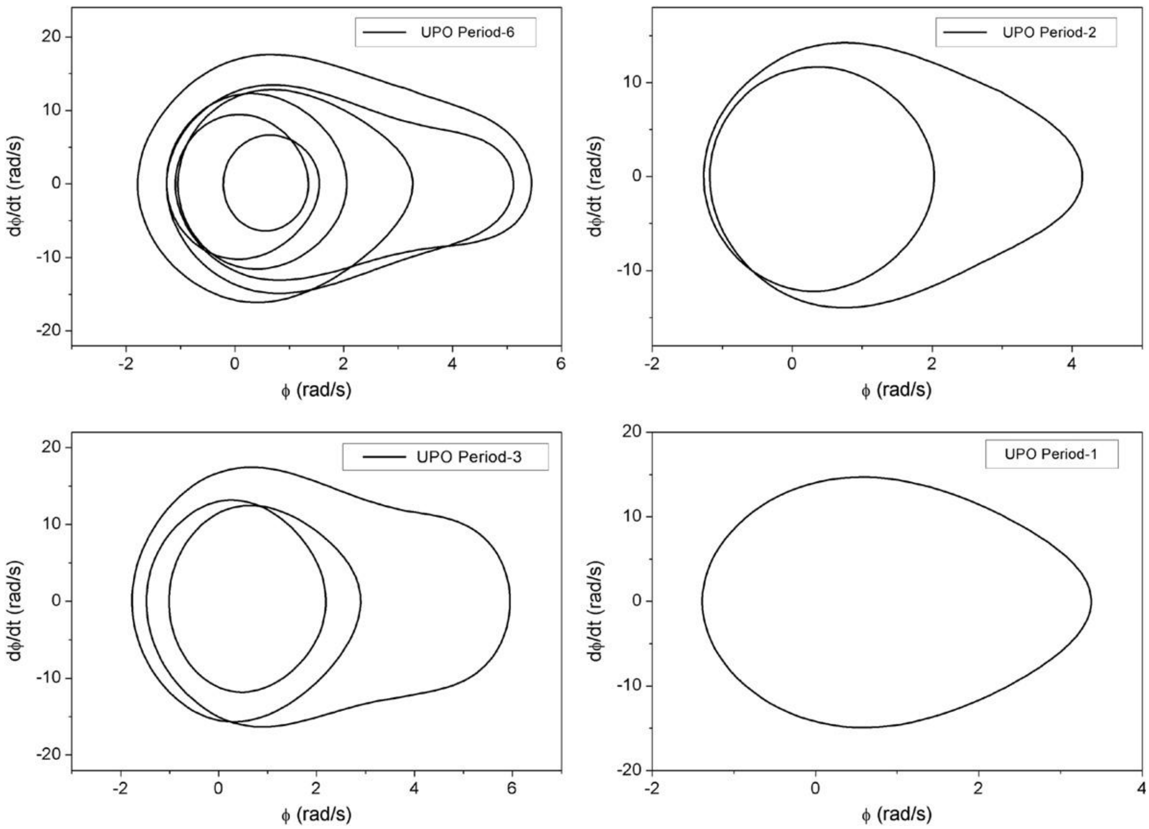

Fig. 17. UPOs of the second control rule.

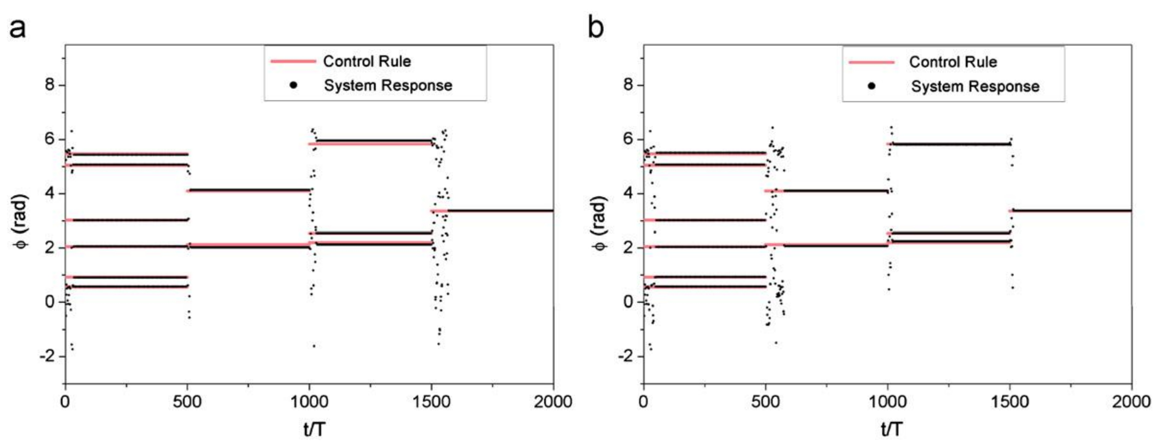

Fig. 18. SC evaluation at the control station #1 with $\eta = 0\%$: (a) parameter Δl_1 and (b) parameter Δl_2.

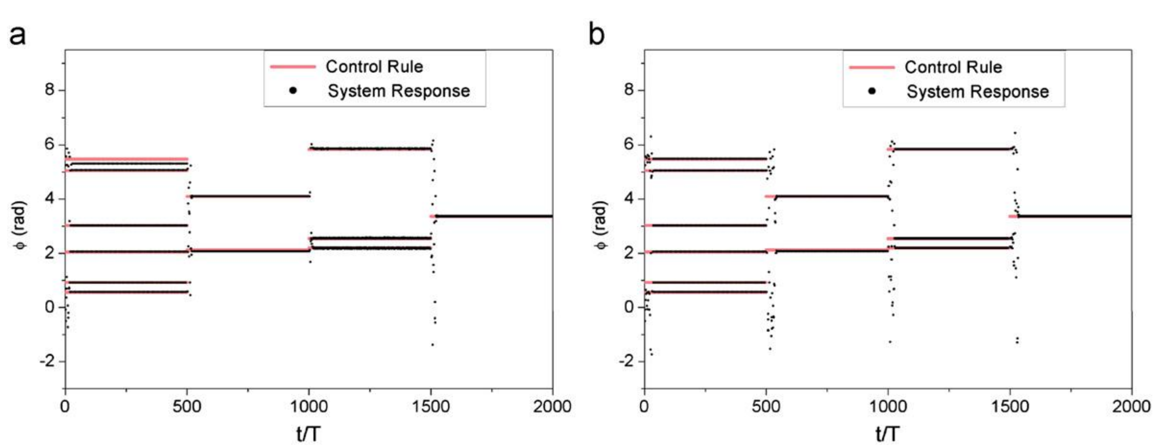

Fig. 19. SC-MP evaluation at the control station #1 with $\eta = 0\%$: (a) coupled and (b) uncoupled approaches.

stabilized by all control methods for an ideal signal: a period-6 orbit during the first 500 periods, a period-2 from period 500 to 1000, a period-3 from 1000 to 1500, and finally a period-1, from period 1500 to 2000. Fig. 16 presents these four UPOs in one of the control stations considered by the semi-continuous methods, while Fig. 17 shows the UPOs in phase space.

Initially, it is considered signals without noise, $\eta=0\%$. Fig. 18 shows the desired trajectory, imposed by the control rule, and the system time evolution at control station #1 when the SC is employed considering the isolated actuation performed by the parameters Δl_1 and Δl_2. Fig. 19 presents the same pictures for the SC-MP, coupled and uncoupled approaches, while Fig. 20 presents results for the ETDF. All methods are able to stabilize all orbits of the control rule. It should be highlighted, however, that the ETDF stabilizes a different UPO for the first orbit of the control rule.

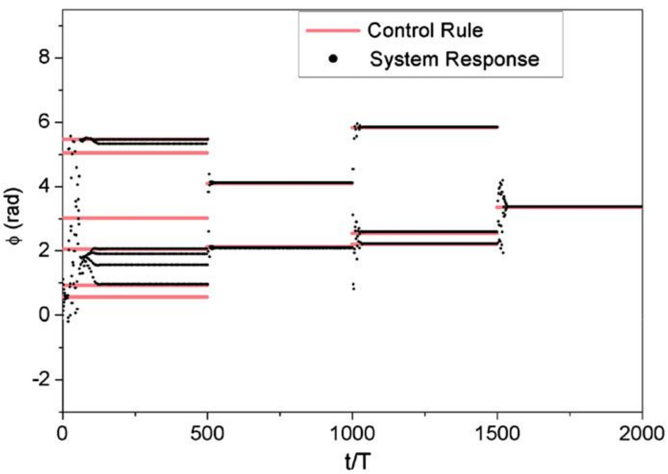

Fig. 20. ETDF evaluation at the control station #1 with $\eta=0\%$.

A noisy signal with 1% of amplitude is now in focus. Fig. 21 shows the desired trajectory, imposed by the control rule, and the system time evolution at control station #1 when the SC is employed considering the isolated actuation performed by the parameters Δl_1 and Δl_2. Fig. 22 presents the same pictures for the SC-MP, coupled and uncoupled approaches, while Fig. 23 presents results for the ETDF. Note that for $\eta=1\%$, the SC with first control parameter stabilizes all UPOs of the control rule, however, sometimes system trajectory escapes from the desired orbit, returning back later. By using the second control parameter, only two of the orbits are successfully stabilized. By using the SC-MP coupled approach, the second orbit of the control rule is not satisfactory stabilized. The uncoupled approach of the SC-MP and the ETDF successfully stabilizes all orbits.

A noise level of 2% is now considered. Fig. 24 shows the desired trajectory imposed by the control rule and the system time evolution at control station #1 when the SC is employed considering the isolated actuation performed by the parameters Δl_1 and Δl_2. Fig. 25 presents the same pictures for the SC-MP, coupled and uncoupled approaches, while Fig. 26 presents results of the

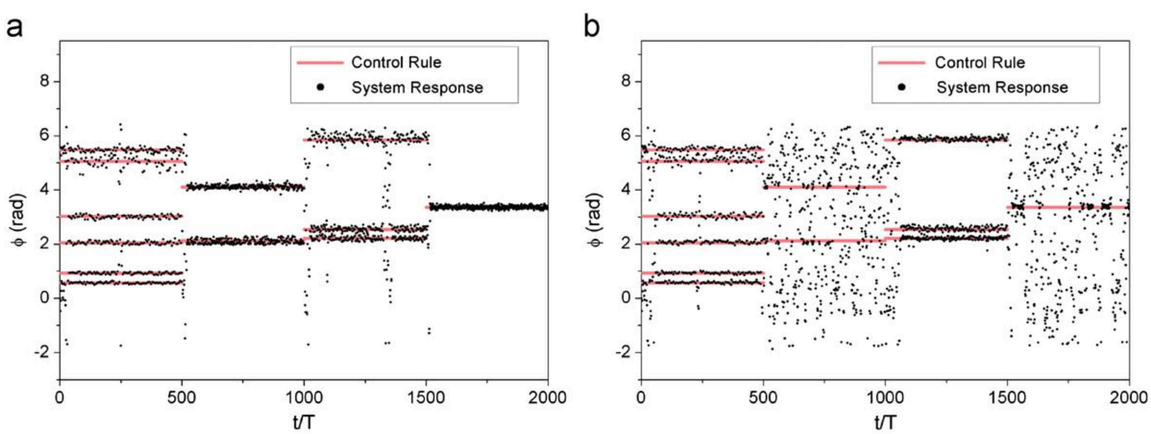

Fig. 21. System controlled using SC at the control station #1 with $\eta=1\%$: (a) parameter Δl_1 and (b) parameter Δl_2.

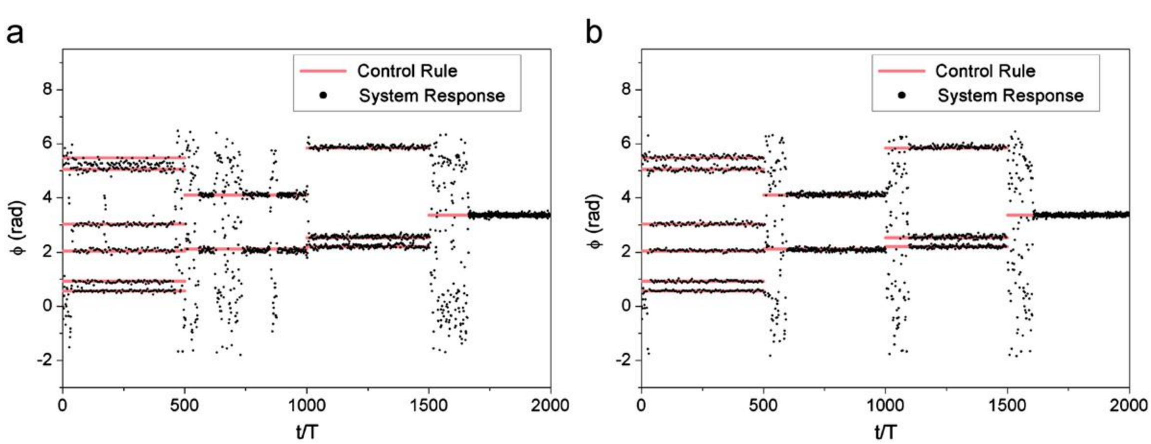

Fig. 22. System controlled using SC-MP at the control station #1 with $\eta=1\%$: (a) coupled and (b) uncoupled approaches.

ETDF. Note that the increase in noise level makes the single-parameter SC to be not able to stabilize some orbits. Although the coupled SC-MP presents better results, it is noticeable that its efficacy decreases with the noise level increase. The uncoupled SC-MP presents better results when compared with the preceding methods and the ETDF successfully stabilize all UPOs of the control rule, except for the fact that the period-6 stabilized orbit is different from the desired one.

Concerning the semi-continuous methods, it should be highlighted that the increase of control stations is a useful procedure in order to avoid the effect of noise, however, the effectiveness of this procedure is limited by the response time of the system [28]. Figs. 27 and 28 present results of the SC with parameter Δl_1 considering four and six control stations for different noise levels ($\eta = 1\%$ and $\eta = 2\%$, respectively). Although the increase of control stations can promote a better performance related to orbit

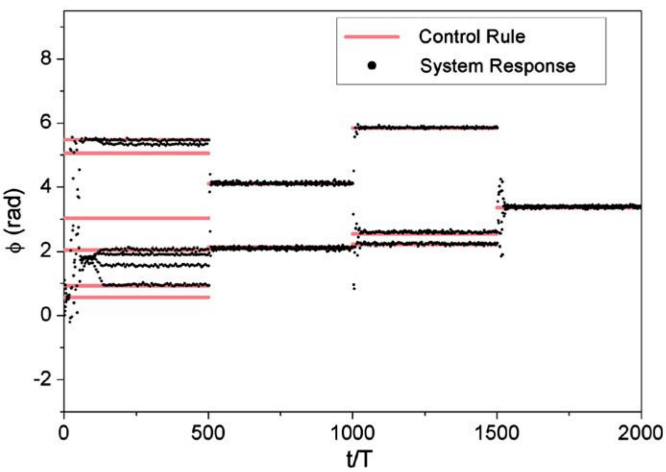

Fig. 23. System controlled using ETDF at the control station #1 with $\eta = 1\%$.

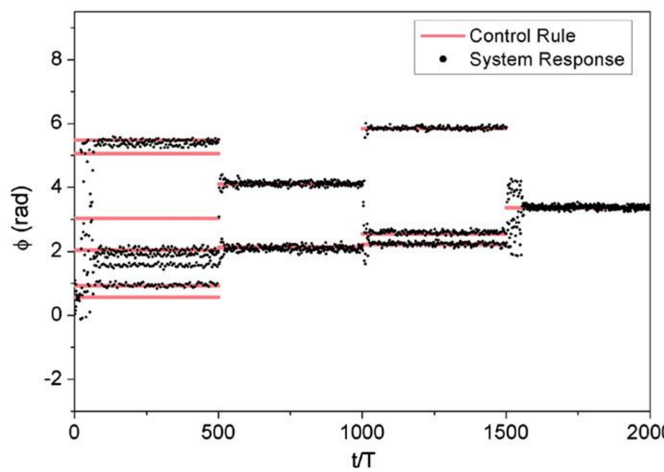

Fig. 26. ETDF evaluation at the control station #1 with $\eta = 2\%$.

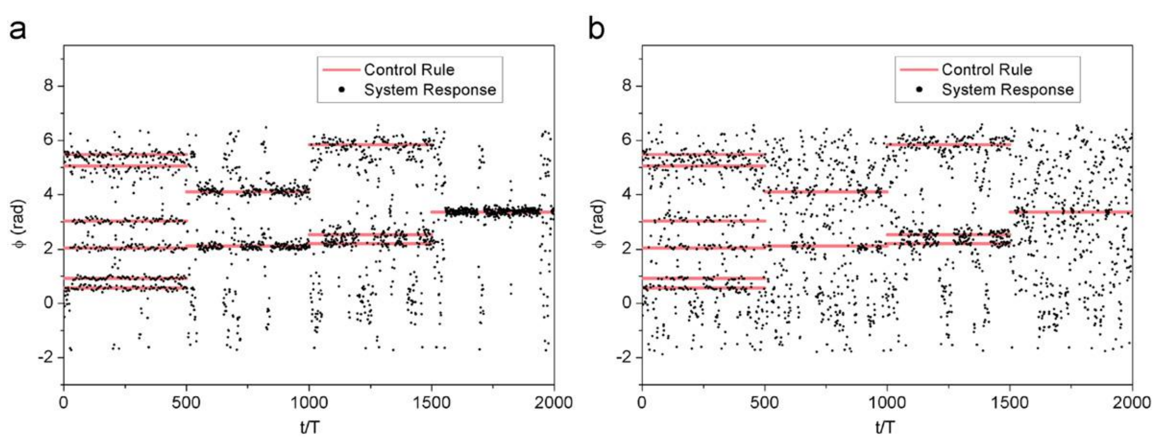

Fig. 24. SC evaluation at the control station #1 with $\eta = 2\%$: (a) parameter Δl_1 and (b) parameter Δl_2.

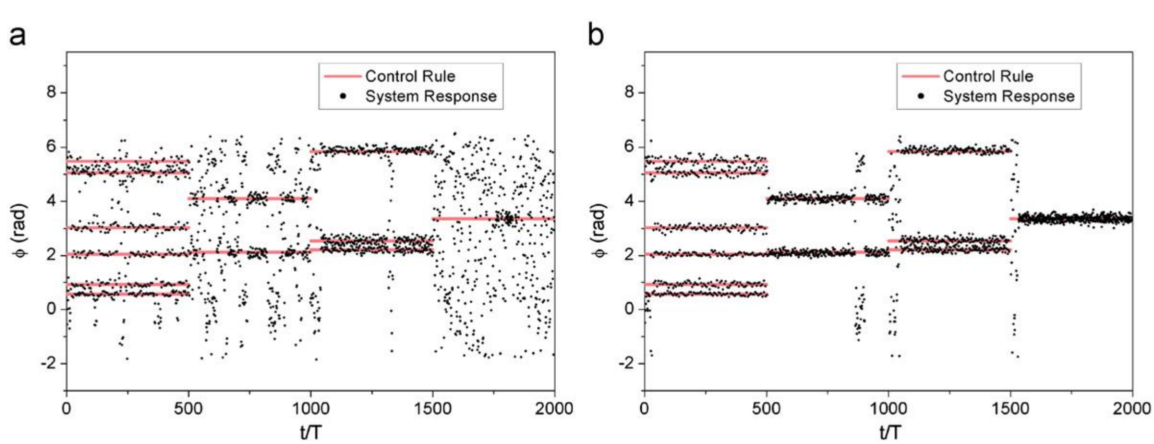

Fig. 25. SC-MP evaluation at the control station #1 with $\eta = 2\%$: (a) coupled and (b) uncoupled approaches.

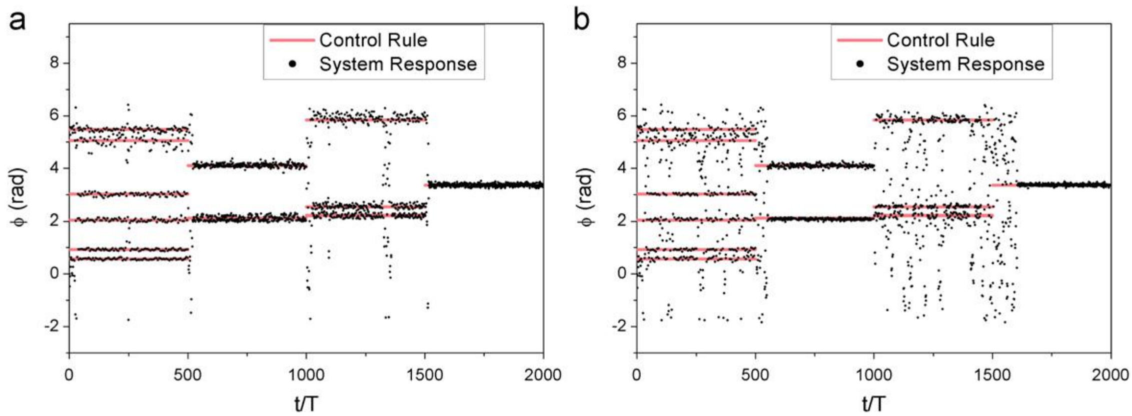

Fig. 27. SC evaluation at the control station #1 with $\eta = 1\%$ and parameter Δl_1: (a) four control stations and (b) six control stations.

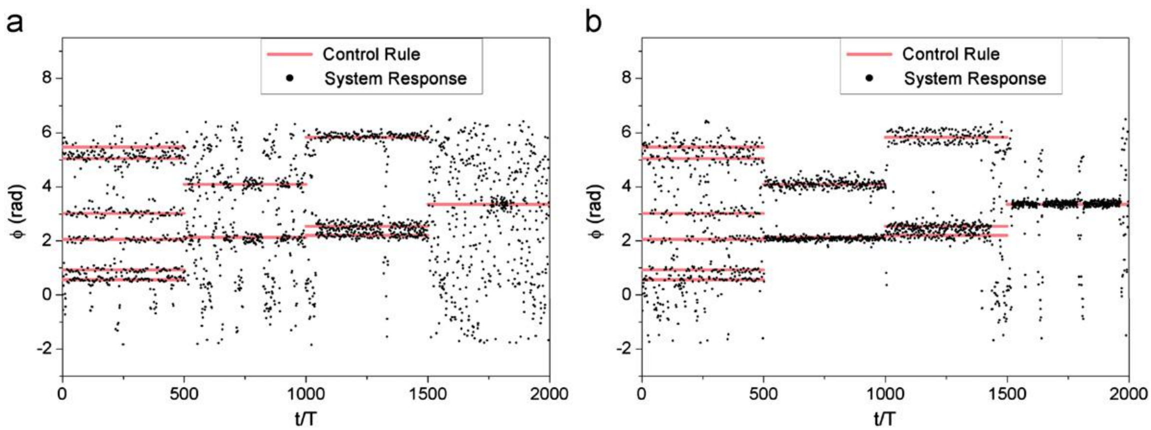

Fig. 28. SC evaluation at the control station #1 with $\eta = 2\%$ and parameter Δl_1: (a) four control stations and (b) six control stations.

stabilization, there are situations where this increase causes the increase of uncertainty that could appear as a consequence of the determination of controller parameters.

For noise levels greater than 2% none of the semi-continuous methods presented good results in stabilizing the non-linear pendulum. The ETDF successfully stabilized orbits of the control rules for noise levels up to $\eta = 5\%$, showing its robustness.

4. Conclusions

This paper presents a comparative analysis of chaos control methods performances. Initially, it is presented an overview of chaos control methods classified as follows: OGY methods – include discrete and semi-continuous approaches; multiparameter methods (MP) – also include discrete and semi-continuous approaches; and time-delayed feedback methods (ETDF) that are continuous approaches. The learning stage is the same for all discrete methods, where system parameters are identified from time series and it is not necessary to know the system dynamics. On the other hand, the learning stage of the continuous methods implies the determination of controller parameters from estimating the maximum Lyapunov exponent, which imposes the knowledge of the mathematical model. In general, systems with high instability need a greater number of actuations which makes the semi-continuous and continuous methods more effective for chaos control. By defining efficacy as the capability to stabilize desired orbits, the semi-continuous methods are more effective than continuous methods to perform system stabilization. The MP coupled approach presents the greatest efficacy between the analyzed methods. The MP uncoupled approach also presents good performance and presents the advantage to avoid the determination of β's parameters when compared to the coupled approach. The continuous methods present low efficacy since it is able to stabilize only few UPOs but avoid the wait time (until the system trajectory falls in the neighborhoods of one UPO fixed point) necessary in the case of discrete methods. However, it should be pointed out that the consideration of this wait time for the continuous method can improve their efficacy. Moreover, continuous methods present a difficulty for the stabilization of high periodicity UPOs since different orbits can be stabilized instead of the desired one. Results from comparative analysis point that the semi-continuous methods present good performance for ideal time series, free of noise. In this regard, it should be highlighted the good performance of the multiparameter approach. When noisy time series is of concern, continuous methods present greater robustness being associated with better performances, however, the uncoupled approach of the semi-continuous multiparameter method also presents a good performance, maintaining other characteristics mentioned before.

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